Institution of Education
"Belarusian State Technological University"
Department of Theoretical Mechanics

Lecture notes on discipline
Applied Mechanics
Part 1. The Theoretical Mechanics

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LECTURE 1
INTRODUCTION TO STATICS.
EQUILIBRIUM OF A SYSTEM OF CONCURRENT FORCE

The science which treats of the general laws of motion and equilibrium of material bodies and of the resulting mutual interactions is called *theoretical*, or *general*, *mechanics*. Theoretical mechanics constitutes one of the scientific bedrocks of modern engineering.

By motion in mechanics we mean mechanical motion, i.e., any change in the relative positions of material bodies in space which occurs in the course of time. By mechanical interaction between bodies is meant such reciprocal action which changes or tends to change the state of motion or the shape of the bodies involved (deformation). The physical measure of such mechanical interaction is called *force*.

Theoretical mechanics is primarily concerned with the general laws of motion and equilibrium of material bodies under the action of the forces to which they are subjected.

According to the nature of the problems treated, mechanics is divided into *statics*, *kinematics* and *dynamics*. Statics studies the forces and the conditions of equilibrium of material bodies subjected to the action of forces. Kinematics deals with the general geometrical properties of the motion of bodies. Finally, dynamics studies the laws of motion of material bodies under the action of forces.

1. The subject of statics

Statics is the branch of mechanics which studies the laws of composition of forces and the conditions of equilibrium of material bodies under the action of forces.

The state of equilibrium or motion of a given body depends on its mechanical interactions with other bodies, i.e., on the loads, attractions or repulsions it experience as a result of such interactions. *In mechanics, the quantitative measure of the mechanical interaction of material bodies is called force.*

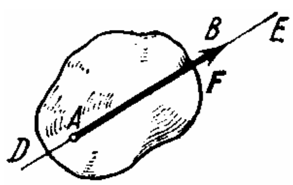


Fig. 1.1

Force is a vector quantity. Its action on a body is characterized by its (1) *magnitude*, (2) *direction*, and (3) *point of application*.

The line *DE* along which the force is directed is called the *line of action* of the force (see Fig. 1.1).

We shall call any set of forces acting on a rigid body a *force system*. We shall also use the following definitions:

1. A body not connected with other bodies and which from any given position can be displaced in any direction in space is called a *free body*.

2. If a force system acting on a free rigid body can be replaced by another force system without disturbing the body's initial condition of rest or motion, the two systems are said to be *equivalent*.

3. If a free rigid body can remain at rest under the action of a force system, that system is said to be *balanced* or *equivalent to zero*.

4. If a given force system is equivalent to a single force, that force is the *resultant* of the system. Thus, *a resultant is a single force capable of replacing the action of a system of forces of on a rigid body*.

A force equal in magnitude, collinear with, and opposite in direction to the resultant is called an *equilibrant* force.

5. Forces acting on rigid body can be divided into two groups: the external forces and the internal forces. *External forces* represent the action of other material bodies on the particles of a given body. *Internal forces* are those with which the particles of a given body act on each other.

6. A force applied to one point of body is called a *concentrated force*. Forces acting on the points of a given volume or given area of a body are called *distributed forces*.

2. Fundamental principles

1st Principle. A free rigid body subjected to the action of two forces can be equilibrium if, and only if, the two forces are equal in magnitude (F_1=F_2), collinear, and opposite in direction. (Fig. 1.2)

2nd Principle. The action of a given force system on a rigid body remains unchanged if another balanced force system is added to, or subtracted from, the original system.

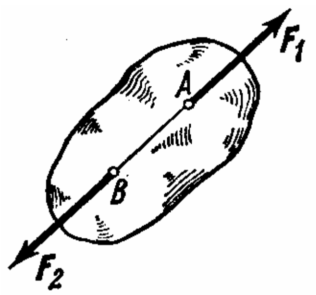


Fig. 1.2

Corollary of the 1st and 2nd Principles. The point of application of a force acting on a rigid body can be transferred to any other point on the line of action of the force without altering its effect.

Thus, the vector denoting force F can be regarded as applied at any point along the line of action (such a vector is called a *sliding vector*).

3rd Principle (the **Parallelogram Law**). Two forces applied at one point of a body have as their resultant a force applied at the same point and represented by the diagonal of parallelogram constructed with the two given forces as its sides.

4th Principle. To any action of one material body on another there is always an equal and oppositely directed reaction. (Fig. 1.3)

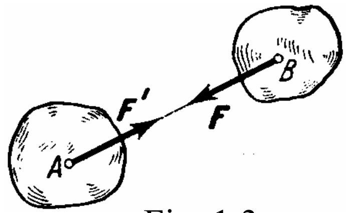


Fig. 1.3

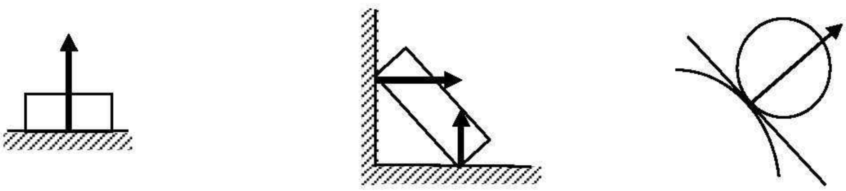
5th Principle. (**Principle of Solidification**). If a freely deformable body subjected to the action of a force system is in equilibrium, the state of equilibrium will not be disturbed if the body solidities (becomes rigid).

3. Constraints and their reactions

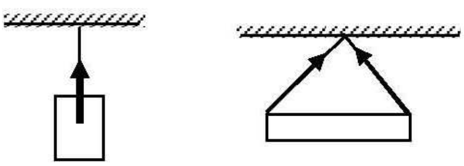
As has been defined above, a body not connected with other bodies and capable of displacement in any direction is called a free body (e.g., a balloon floating air). A body whose displacement in space is restricted by other bodies, either connected to or in contact with it, is called a *constrained body*. *We shall call a constraint anything that restricts the displacement of a given body in space*.

A body acted upon by a force or forces whose displacement is restricted by a constraint acts on that constraint with a force which is customarily called the load or pressure acting on the constraint. At the same time, according to the 4th principle, the constraint reacts with a force of same magnitude and opposite sense. *The force with which a constraint acts on a body, thereby restricting its displacement, is called the force of reaction of the constraint* (*force of constraint*) or simply *the reaction of the constraint*.

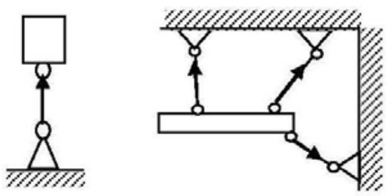
Some common types of constraints:



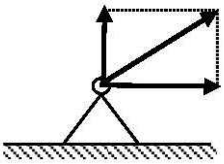
1. Smooth plane (surface) or support



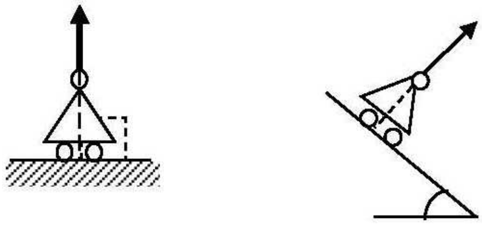
2. String



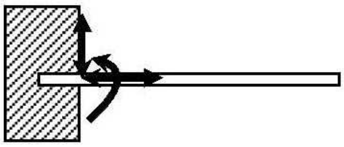
3. Pin-type rod



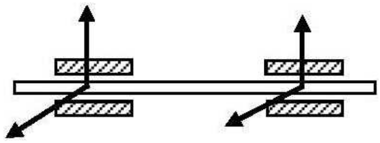
4. Fixed pin



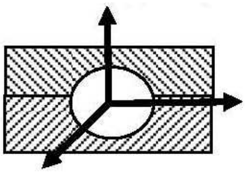
5. Pin or roller support



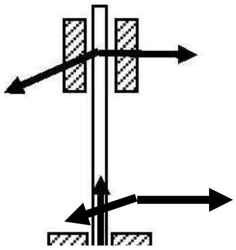
6. Fixed support (rigid clamp or embedding)



7. Cylindrical pin



8. Ball-and-socket joint



9. Step bearing

4. Equilibrium of a system of concurrent force

For a system of concurrent forces acting on a body to be in equilibrium it is necessary and sufficient for the resultant of the forces to be zero. The conditions which the forces themselves must satisfy can be expressed either in graphical or in analytical form.

Graphical Condition of Equilibrium. Since the resultant **R** of a system of concurrent forces is defined as the closing side of a force polygon constructed with the given forces, it follows that **R** can be zero only if the terminal point of the last force of the polygon coincides with the initial point of the first force. i.e., if polygon is closed.

Thus, *for a system of concurrent forces to be in equilibrium it in necessary and sufficient for the force polygon drawn with these forces to be closed.*

Analytical Conditions of Equilibrium. Analytically the resultant of a system of concurrent forces is determined by the formula

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad (1.1)$$

As the expression under the radical is a sum of positive components, R can be zero only if simultaneously R_x=0, R_y=0, R_z=0, i.e., when the forces acting on the body satisfy the equations

$$\sum_k F_{kx} = 0, \quad \sum_k F_{ky} = 0, \quad \sum_k F_{kz} = 0. \qquad (1.2)$$

This equation express the conditions of equilibrium in analytical form: *The necessary and sufficient condition for the equilibrium of a three-dimensional system of concurrent forces is that the sums of the projections of all the forces on each of three coordinate axes must separately vanish.*

If all the concurrent forces acting on a body lie in one plane, they form a *coplanar system of concurrent forces*. Obviously, for such a force system only two equations are required to express the conditions of equilibrium:

$$\sum_k F_{kx} = 0, \quad \sum_k F_{ky} = 0. \qquad (1.3)$$

Eng. (2) and (3) also express the necessary conditions (or equations) of equilibrium of a free rigid body subjected to the action of concurrent forces.

The Theorem of Three Forces. The following theorem will often be found useful in solving problems of statics: *If a free rigid body remains in equilibrium under the action of three nonparallel coplanar forces, the lines of action of those forces intersect at one point.*

LECTURE 2
CONDITIONS FOR THE EQUILIBRIUM OF A COPLANAR FORCE SYSTEM
1. Moment of force about a point

Consider a force **F** applied at a point A of a rigid body (Fig. 2.1) which tends to rotate the body about a point O. The perpendicular distance h from O to the line of action of **F** is called the *moment arm* of force **F** about the centre O.

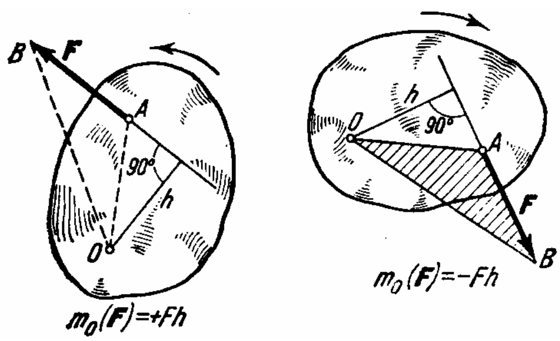


Fig. 2.1

The moment of a force **F**

about a centre O is defined as the product of the force magnitude and the length of the moment arm taken with appropriate sign.

We shall denote the moment of a force **F** about a centre O by the symbol $m_0(\mathbf{F})$. Thus,

$$m_O(\mathbf{F}) = \pm Fh. \tag{2.1}$$

We shall call a moment positive if the applied force tends to rotate the body counterclockwise, and negative if it tends to rotate the body clockwise. Thus, the sign of the moment of the force **F** about O is (+) in Fig. 2.1*a*, and (–) in Fig. 2.1*b*. If the arm is measured in metres, the moment of the force is measured in newton-metres (Nm).

Note the following properties of the moment of the force:

(1) The moment of a force does not change if the point of application of the force is transferred along its line of action.

(2) The moment of force about a centre O can be zero only if the force is zero or if its line of action passes through O (i.e., the moment arm is zero).

Varignon′s Theorem of the Moment of a Resultant. *The moment of the resultant of a coplanar system of concurrent forces about any centre is equal to the algebraic sum of the moments of the component forces about that centre.*

2. A Force Couple. Moment of a Couple

A *force couple* is a system of two parallel forces of same magnitude and opposite sense acting on a rigid body (Fig. 2.2). Thus, a couple cannot be replaced or balanced by a single force. For this reason the properties of the couple as a special mode of mechanical interaction between bodies are the subject of a special study.

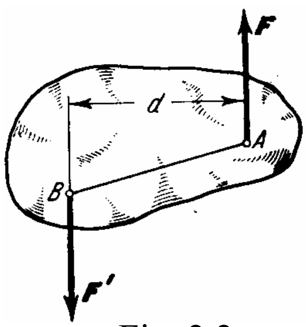


Fig. 2.2

The plane through the lines of action of both forces of a couple is called the *plane of action of the couple*. The perpendicular distance d between the lines of action of the forces is called the *arm of the couple*. A couple is characterized by its *moment*.

For this case the following definition can be given in analogy with that of the moment of a force: *The moment of a couple is defined as a quantity equal to the product of the magnitude of one of the forces of the couple and* *the perpendicular distance between the forces, taken with the appropriate sign*. Denoting the moment of a couple by the symbol *m* or *M*, have:

$$m_O(\mathbf{F}) = \pm Fd\,. \tag{2.2}$$

The moment of a couple (as that of a force) *is said to be positive if the action of the couple tends to turn a body counterclockwise, and negative if clockwise*.

Let us prove the following **theorem of the moments of the forces of a couple**: *The algebraic sum of the moments of the forces of a couple about any point in its plane of action is independent of the location of that point and is equal to the moment of the couple*.

Before stating the conditions necessary for two couples to be equivalent let us prove the following theorem: *A couple acting on a rigid body can be replaced by any other couple of the same moment lying in the same plane without altering the external effect on that body*.

3. Theorem of translation of a force

A force acting on a rigid body can be moved parallel to its line of action to any point of the body, if we add a couple of a moment equal to the moment of the force about the point to which it is translated.

Consider a force **F** applied to a rigid body at a point *A* (Fig. 2.3). The action of the force will not change if two balanced forces $\mathbf{F}'=\mathbf{F}$ and $\mathbf{F}''=-\mathbf{F}$ are applied at any point *B* of the body. The resulting three-force system consists of a force $\mathbf{F}'$, equal to **F** and applied at *B*, and a couple $(\mathbf{F}'',-\mathbf{F})$ of moment

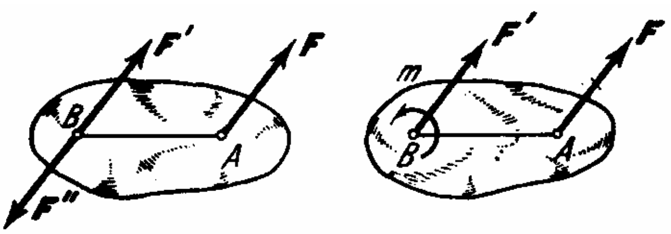


Fig. 2.3

$$m = m_B(\mathbf{F})\,. \tag{2.3}$$

Corollary of the theorem of translation of a force. *Any system of coplanar forces acting on a rigid body can be reduced to an arbitrary centre O in such a way that it is replaced by a single force* **R** *equal to the principal vector of the system and applied at the centre of reduction O and a single couple of moment* M_O *equal to the principal moment of the system about O.*

4. Conditions for the equilibrium of a coplanar force system

For any given coplanar force system to be in equilibrium it is necessary and sufficient for the following two conditions to be satisfied simultaneously:

$$\mathbf{R} = 0, \quad M_O = 0, \tag{2.4}$$

where O is any point in a given plane, as at $\mathbf{R}$=0 the magnitude of M_O does not depend on the location of O.

1. **The Basic Equations of Equilibrium.** The magnitudes of $\mathbf{R}$ and M_O are determined by the equations

$$R = \sqrt{R_x^2 + R_y^2}, \quad M_O = \sum m_O(\mathbf{F}_k), \tag{2.5}$$

where

$$R_x = \sum_k F_{kx} \text{ and } R_y = \sum_k F_{ky} \tag{2.6}$$

But R can be zero only if both R_x=0 and R_y=0. Hence, Eqs. (2.5) will be satisfied if

$$\sum_k F_{kx} = 0, \quad \sum_k F_{ky} = 0, \quad \sum_k m_O(\mathbf{F}_k) = 0. \tag{2.7}$$

Eqs. (2.7) express the following analytical conditions of equilibrium: *The necessary and sufficient conditions for the equilibrium of any coplanar force system are that the sums of the projections of all the forces on each of the two coordinate axes and the sum of the moments of all the forces about any point in the plane must separately vanish.*

2. The Second Form of the Equations of Equilibrium: *The necessary and sufficient conditions for the equilibrium of any coplanar force system are that the sums of the moments of all the forces about any two points A and B and the sum of the projections of all the forces on any axis Ox not perpendicular to AB must separately vanish:*

$$\sum_k F_{kx} = 0, \quad \sum_k m_A(\mathbf{F}_k) = 0, \quad \sum_k m_B(\mathbf{F}_k) = 0. \tag{2.8}$$

3. The Third Form of the Equations of Equilibrium (the Equations of Three Moments): *The necessary and sufficient conditions for the equilibrium of any coplanar force system are that the sums of the moments of all the forces about any three non-collinear points A, B, C must separately vanish:*

$$\sum_k m_A(\mathbf{F}_k) = 0, \quad \sum_k m_B(\mathbf{F}_k) = 0, \quad \sum_k m_C(\mathbf{F}_k) = 0\,. \qquad (2.9)$$

5. Equilibrium of a coplanar system of parallel force

If all the forces acting on a body are parallel (Fig. 2.4), we can take axis x of a coordinate system perpendicular to them and axis y parallel to them. Then the x projections of all the forces will be zero, and the first one of Eqs. (2.8) becomes an identity 0 = 0. Hence, for parallel forces we have two equations of equilibrium:

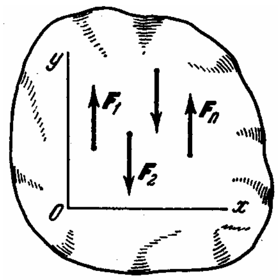


Fig. 2.4

$$\sum_k F_{ky} = 0, \quad \sum_k m_O(\mathbf{F}_k) = 0\,, \qquad (2.10)$$

where the y axis is parallel to the forces.

Another form of the conditions for the equilibrium of parallel forces derived from Eqs. (2.9) is:

$$\sum_k m_A(\mathbf{F}_k) = 0, \quad \sum_k m_B(\mathbf{F}_k) = 0\,. \qquad (2.11)$$

The points A and B should not lie on a straight line parallel to the given forces.

LECTURE 3
EQUILIBRIUM OF SYSTEMS OF BODIES
1. Distributed forces

In engineering problems we often have to deal with loads distributed over an area according to a known mathematical law. Let us examine some simple cases of distributed coplanar forces.

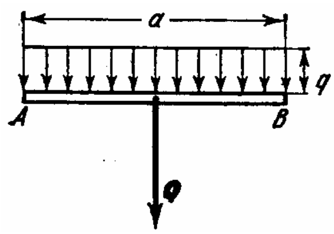


Fig. 3.1

A plane system of distributed forces is characterized by the load per unit length of the line of application, which is called the *intensity* q. The dimension of intensity is newtons per metre (N/m).

(1) Forces Uniformly Distributed Along a Straight Line (Fig. 3.1). The intensity q of such a system is a constant quantity. In solving problems of statics such a force system can be replaced by its resultant $\mathbf{Q}$ of magnitude

$$Q = aq\,. \qquad (3.1)$$

applied at the middle of AB.

(2) Forces Distributed Along a Straight Line According to a Linear Law (Fig. 3.2). An example of such a load is the pressure of water against a dam, which drops from a maximum at the bottom to zero at the surface. For such forces the intensity q varies from zero to q_m. The resultant $\boldsymbol{Q}$ is determined in the same manner as the resultant of the gravity forces acting on a homogeneous triangular lamina ABC. As the weight of a homogeneous lamina is proportional to its area, the magnitude of $\boldsymbol{Q}$ is

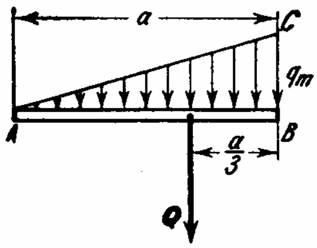


Fig. 3.2

$$Q = \frac{1}{2} a q_m . \qquad (3.2)$$

and is applied at a point at a distance of $a/3$ from side BC of triangle ABC.

2. Problems statically determinate and statically indeterminate

In problems where the equilibrium of constrained rigid bodies is considered, the reactions of the constraints are unknown quantities. Their number depends on the number and type of the constraints. A problem of statics can be solved only if the number of unknown reactions is not greater than number of equilibrium equations in which they are present. Such problem are called *statically determinate*, and the corresponding systems of bodies are called *statically determinate systems*.

Problems in which the number of unknown reactions of the constraints is greater than number of equilibrium equations in which they are called *statically indeterminate*, and the corresponding systems of bodies are called *statically indeterminate systems*.

3. Equilibrium of systems of bodies

In many cases the static solution of engineering structures is reduced to an investigation of the conditions for the equilibrium of systems of connected bodies. We shall call the constraints connecting the parts of a given structure internal, as opposed to external constraints which connect a given structure with other bodies (e. g., the supports of a bridge).

If a structure remains rigid after the external constraints (supports) are removed, the problems of statics are solved for it as for a rigid body.

However, an engineering structure may not necessarily remain rigid when the external constraints are removed. An example of such a structure is the three-pin arch in Fig. 3.3. If supports A and B are removed the arch is no longer rigid, for its parts can turn about pin C.

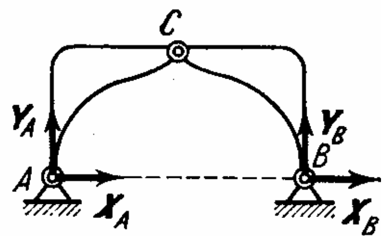


Fig. 3.3

According to the principle of solidification, for a system of forces acting on such a structure to be in equilibrium it must satisfy the conditions of equilibrium for a rigid body. It was pointed out, though, that these conditions, while necessary, were not sufficient, and therefore not all the unknown quantities could be determined from them. In order to solve such a problem it is necessary to examine additionally the equilibrium of one or several parts on the given structure.

For example, for the forces acting on the three-pin arch in Fig. 68 we have three equations with four unknown quantities, X_A, Y_A, X_B, Y_B. By investigating the conditions for the equilibrium of the left- or right-hand members of the arch we obtain three more equations with two more unknown quantities, X_C and Y_C (not shown in Fig. 3.3). Solving the system of six equations we can determine all six unknown quantities.

Another method of solving such problems is to divide a structure into separate bodies and write the equilibrium equations for each as for a free body. The reactions of the internal constraints will constitute pairs of forces equal in magnitude and opposite in sense. For a structure of n bodies, each of which is subjected to the action of a coplanar force system, we thus have $3n$ equations from which we may determine $3n$ unknown quantities (in other force systems the number of equations is, of course, different). If the number of unknown quantities is greater than the number of equations, the problem is statically indeterminate.

LECTURE 4
FRICTION
1. Laws of static friction

We know from experience that when two bodies tend to slide on each other, a resisting force appears at their surface of contact which opposes their relative motion. This force is called sliding friction.

Friction is due primarily to minute irregularities on the contacting surfaces, which resist their relative motion, and to forces of adhesion between contacting surfaces. A detailed examination of the nature of friction is a complex physic-mechanical problem lying beyond the scope of theoretical mechanics.

Engineering calculations are based on several general laws deduced from experimental evidence, which reflect the principal features of friction with an accuracy sufficient for practical purposes. These laws, the laws of sliding friction, can be formulated as follows:

(1) When two bodies tend to slide on each other, a frictional force is developed at the surface of contact, the magnitude of which can have any value from zero to a maximum value F_l which is called *limiting friction*, or *friction of impending motion*.

Frictional force is opposite in direction to the force which tends to move a body.

(2) Limiting friction is equal in magnitude to the product of the coefficient of static friction (or friction of rest) f_0 and the normal pressure or normal reaction N:

$$F_l = f_0 N . \tag{4.1}$$

The coefficient of static friction f_0 is a dimensionless quantity which is determined experimentally and depends on the material of the contacting bodies and the conditions of the surfaces (their finish, temperature, humidity, lubrication, etc.).

(3) Within fairly broad limits, the value of limiting friction does not depend on the area of the surface of contact.

Taken together, the first and second laws state that for conditions of equilibrium the static friction (adhesive force) $F \le F_l$ or

$$F \le f_0 N . \tag{4.2}$$

The following table offers an idea of the values of the coefficient of static friction for various materials:

Wood on wood…………0.4 to 0.7
Metal on metal…………0.15 to 0.25
Steel on ice……………..0.027

For more detailed information the student is invited to consult engineering hand books.

The foregoing refers to sliding friction of rest. When motion occurs, the frictional force is directed opposite to the motion and equals the product of the coefficient of kinetic, or sliding, friction and the normal pressure:

$$F = fN . \tag{4.3}$$

The coefficient of kinetic friction f is also a dimensionless quantity which is determined experimentally. The value of f depends not only on the material and conditions of the contacting surfaces but also, to some degree, on the relative velocity of the bodies. In most cases the value of f at first decreases with velocity and then attains a practically constant value.

2. Reactions of Rough Constraints. Angle of Friction

Up till now, in solving problems of statics, we neglected friction and regarded the surfaces of constraints as smooth and their reactions as normal to the surface. The reactions of real (rough) constraints consist of two components: the normal reaction **N** and the frictional force **F** perpendicular to it. Consequently, the total reaction **R** forms an angle with the normal to the surface. As the friction increases from zero to $\mathbf{F}_l$ force **R** changes from **N** to $\mathbf{R}_l$ its angle with the normal increasing from zero to a maximum value φ_0 (Fig. 4.1). The maximum angle φ_0 which the total reaction of a rough support makes with the normal to the surface is called the *angle of static friction*, or *angle of repose*.

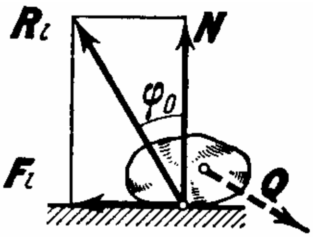


Fig. 4.1

From the diagram we have:

$$\operatorname{tg} \varphi_0 = \frac{F_l}{N}. \tag{4.4}$$

Since $F_l = f_0 N$, we have the following relation between the angle of friction and the coefficient of friction:

$$\operatorname{tg} \varphi_0 = f_0. \tag{4.5}$$

When a system is in equilibrium the total reaction **R** can pass anywhere within the angle of friction, depending on the applied forces. When motion impends, the angle between the reaction and the normal is φ_0.

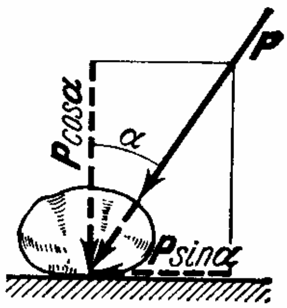


Fig. 4.2

If to a body lying on a rough surface is applied a force **P** making an angle α with the normal (Fig. 4.2), the body will move only if the shearing force **P** $\sin\alpha$ is greater than $F_l = f_0 P\cos\alpha$ (neglecting the weight of the body and considering $N = P\cos\alpha$). But the inequality $P\sin\alpha > f_0 P\cos\alpha$, where $f_0 = \operatorname{tg}\varphi_0$, is satisfied only if $\operatorname{tg}\alpha > \operatorname{tg}\varphi_0$, i.e., if $\alpha > \varphi_0$. Consequently, if angle α is less than φ_0 the body will remain at rest no matter how great the applied force. This explains the well-known phenomena of wedging and self-locking.

3. Equilibrium with friction

Examination of the conditions for the equilibrium of a body taking friction into account is usually limited to a consideration of the conditions when motion is impending and the frictional force acquires its maximum value F_l. For the analytical solution of problems the reaction of a rough constraint is denoted by its two components **N** and $\mathbf{F}_l$, where $F_l = f_0 N$. The known equations of static equilibrium are then written, substituting $f_0 N$ for F_l, and solved for the required values.

If the problem requires that all possible positions of equilibrium be determined, it is sufficient to solve only for the position of impending motion. Other positions of equilibrium can then be found by reducing the coefficient of friction f_0 in the obtained solution to zero.

It is important to note that in positions of equilibrium when motion does not impend the force of friction F is not equal to F_l and its magnitude, if it is required, should be determined from the conditions of equilibrium as a new unknown quantity.

In graphical solutions it is more convenient to denote the reaction of a rough constraint by a single force **R**, which in the position of impending motion will be inclined at an angle φ_0 to the normal to the surface.

5. Rolling friction and pivot friction

Rolling friction is defined as the resistance offered by a surface to a body rolling on it.

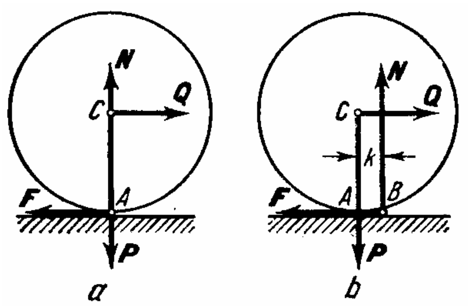


Fig. 4.3

Consider a roller of radius R and weight P resting on a rough horizontal surface (Fig. 4.2*a*). If we apply to the axle of the roller a force $Q < F_l$, there will be developed at A a frictional force **F**, equal in magnitude to Q**,** which prevents the roller from slipping on the surface. If the normal reaction **N** is also assumed to be applied at A**,** it will balance force **P**, with forces **Q** and **F** making a couple which turns the roller. If these assumptions were correct, we could expect the roller to move, howsoever small the force **Q**.

Experience tells us however, that this is not the case; for, due to deformation, the bodies contact over a certain surface AB (Fig. 4.3*b***).** When force **Q** acts, the pressure at A decreases and at B increases. As a result, the reaction **N** is shifted in the direction of the action of force **Q.** As **Q** increases, this displacement grows till it reaches a certain limit *k*. Thus, in the position of impending motion, acting on the roller will be a couple ($\mathbf{Q}_l$,**F**) with a moment Q_lR balanced by a couple (**N**,**P**) of moment *Nk.* As the moments are equal, we have $Q_lR=Nk$, or

$$Q_l = \frac{k}{R} N . \tag{4.6}$$

As long as $Q<Q_l$ the roller remains at rest; when $Q>Q_l$ it starts to roll.

The linear quantity *k* in Eq. (4.6) is called the *coefficient of rolling friction,* or *resistance,* and is generally measured in metres. The value of *k* depends on the material of the bodies and is determined experimentally. The following list offers an idea of some typical values of *k*:

Wood on wood………………………… 0.05 to 0.08 cm

Mild steel on steel (wheel on rail)……... 005 cm

Hardened steel on steel (ball bearing)..... 0.001 cm

The ratio *k*/*R* for most materials is much less than the coefficient of static friction f_0. That is why in mechanisms rolling parts (wheels, rollers, ball bearings, etc.) are preferred to sliding parts.

LECTURE 5
EQUILIBRIUM OF AN ARBITRARY FORCE SYSTEM IN SPACE
1. Moment of a force about a point as a vector

Before proceeding with the solution of problems of statics for force systems in space, we should elaborate some of the concepts introduced in the preceding lectures. Let us begin with the concept of moment of a force.

Thus, *the moment of a force* **F** *about center O is equal to the cross product of the radius vector* **r**=**OA**, *from O to the point of application A of the force, and the force itself.*

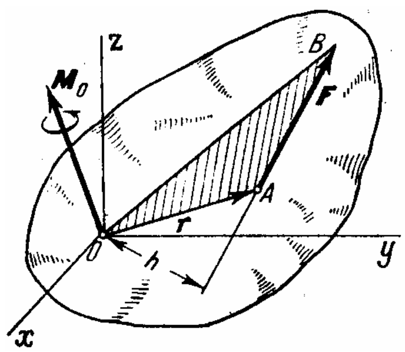


Fig. 5.1

$$\mathbf{M}_O = \mathbf{OA} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} , \tag{5.1}$$

2. Moment of a force with respect to an axis

Before proceeding with the solution of problems of statics for any force system in space we must introduce the concept of moment of a force about an axis.

The moment of a force about an axis is the measure of the tendency of the force to produce rotation about that axis. Consider a rigid body free to rotate about an axis z (Fig. 5.2). Let a force **F** applied at A be acting on the body. Let us now pass a plane xy through point A normal to the axis z and let us resolve the force **F** into rectangular components $\mathbf{F}_z$ parallel to the z-axis and $\mathbf{F}_{xy}$ in the plane xy ($\mathbf{F}_{xy}$ is in fact the projection of force **F** on the plane xy). Obviously, force $\mathbf{F}_z$, being parallel to axis z, cannot turn the body about that axis (it only tends to translate the body *along* it). Thus we find that the total tendency of force **F** to rotate the body is the same as that of its component $\mathbf{F}_{xy}$. We conclude, then, that

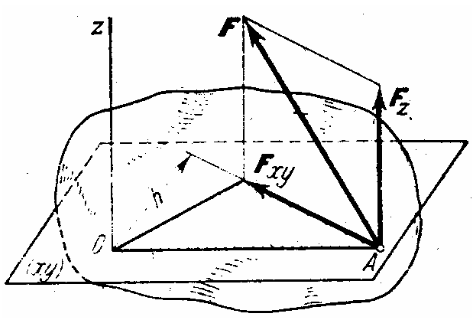


Fig. 5.2

$$m_z(\mathbf{F}) = m_z(\mathbf{F}_{xy}), \tag{5.2}$$

where $m_z(\mathbf{F})$ denotes the moment of force **F** with respect to axis z. But the rotational effect of force $\mathbf{F}_{xy}$, which lies in a plane perpendicular to axis z, is the product of the magnitude of this force and its distance h from the axis. The moment of force $\mathbf{F}_{xy}$ with respect to point O, where the axis pierces the plane xy, is the same. Hence,

$$m_z(\mathbf{F}_{xy}) = m_O(\mathbf{F}_{xy}), \tag{5.3}$$

or, by Eq. (5.2),

$$m_z(\mathbf{F}) = m_O(\mathbf{F}_{xy}) = \pm\mathbf{F}_{xy}h, \tag{5.4}$$

From this we deduce the following definition: *The moment of a force about an axis is an algebraic quantity equal to the moment of the projection of that force on a plane normal to the axis with respect to the point of intersection of the axis and the plane.*

In determining moments the following special cases should be borne in mind:

(1) If a force is parallel to an axis, its moment about that axis is zero (since F_{xy}=0).

(2) If the line of action of a force intersects with the axis, its moment with respect to that axis is zero (since h=0).

Combining the two cases, we conclude that *the moment of a force with respect to an axis is zero if the force and the axis are coplanar*.

(3) If a force is perpendicular to an axis, its moment about that axis is equal to the product of the force magnitude and the perpendicular distance from the force to the axis.

Varignon's Theorem: *if a given force system has a resultant, the moment of that resultant with respect to any axis is equal to the algebraic sum of the moments of the component forces with respect to the same axis*

3. Reduction of a force system in space to a given centre

We have thus proved the following theorem*: Any system of forces acting on a rigid body can be reduced to an arbitrary centre O and replaced by a single force* **R***, equal to the principal vector of the system applied at the centre of reduction, and a couple with a moment* $\mathbf{M}_O$*, equal to the principal moment of the system with respect to O* (Fig. 5.3).

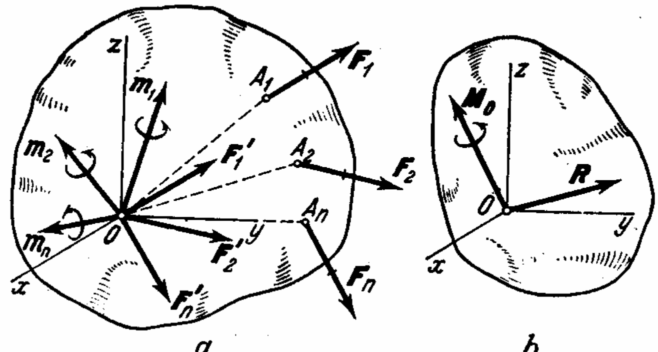


Fig. 5.3

$$\mathbf{R} = \sum_k \mathbf{F}_k, \quad \mathbf{M}_O = \sum_k \mathbf{m}_O(\mathbf{F}_k), \qquad (5.5)$$

4. Conditions of equilibrium of an arbitrary force system in space

Like a coplanar force system, any force system in space can be reduced to a point O and replaced by a resultant force **R** and couple of moment $\mathbf{M}_O$, [the values of **R** and $\mathbf{M}_O$, are determined from Eqs. (5.5)]. Reasoning we come to the conclusion that the necessary and sufficient conditions for the given system of forces to be in equilibrium are that $\mathbf{R}=0$ and $\mathbf{M}_O=0$. But vectors **R** and $\mathbf{M}_O$, can be zero only if all their projections on the coordinate axes are zero, i.e., when $R_x=R_y=R_z=0$ and $M_x=M_y=M_z=0$, or when the acting forces satisfy the conditions

$$\begin{gathered}\sum_k F_{kx} = 0, \quad \sum_k F_{ky} = 0, \quad \sum_k F_{kz} = 0, \\ \sum_k m_x(\mathbf{F}_k) = 0, \quad \sum_k m_y(\mathbf{F}_k) = 0, \quad \sum_k m_z(\mathbf{F}_k) = 0.\end{gathered} \qquad (5.6)$$

Thus*, the necessary and sufficient conditions for the equilibrium of any force system in space are that the sums of the projections of all the forces on each of the three coordinate axes and the sums of the moments of all the forces about those axes must separately vanish.*

5. The case of parallel forces

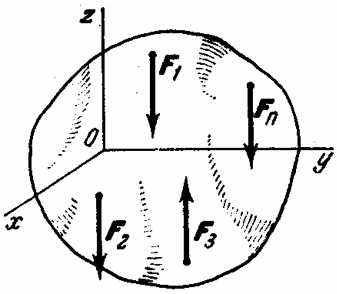


Fig. 5.4

If all the forces acting on a body are parallel, the coordinate axes can be chosen so that the axis z is parallel to the forces (Fig. 5.4). Then the x and y projections of all the forces will be zero, their moments about axis z will be zero, and the Eqs. (5.6) will be reduced to three conditions of equilibrium:

$$\sum_k F_{kz} = 0, \quad \sum_k m_x(\mathbf{F}_k) = 0, \quad \sum_k m_y(\mathbf{F}_k) = 0 \tag{5.7}$$

The other equations will turn into identities 0 = 0.

Thus, the necessary and sufficient conditions for the equilibrium of a system of parallel forces in space are that the sum of the projections of all the forces on the coordinate axis parallel to the forces and, the sums of the moments of all the forces about the other two coordinate axes must separately vanish.

LECTURE 6
KINEMATICS OF A PARTICLE
1. Introduction to kinematics

Kinematics is that part of the theoretical mechanics that deals with the study of the mechanical motion without to consider the forces and the masses of the bodies in motion, namely studies the geometry of the motion. We remind that through mechanical motion we understand the changing the position of bodies (or parts from bodies) with respect to other bodies considered as reference systems.

In the kinematics we shall have to solve generally two problems: to determine the position of the particle (or of the body) in each instant of the motion, and to know how moves the particle (or the body).

For to define the position of the particle we can use the *vector method of describing motion* (used in theoretical demonstrations generally), *coordinate method of describing motion* and *natural method of describing motion*.

For to define how the motion is made we shall introduced two vector notions: *velocity* and *acceleration*.

2. Method of describing motion of a particle

Vector method of describing motion. In the first case is used the *radius vector* **r**, that in absolute motion is represented with respect to a fixed point (Fig. 6.1).

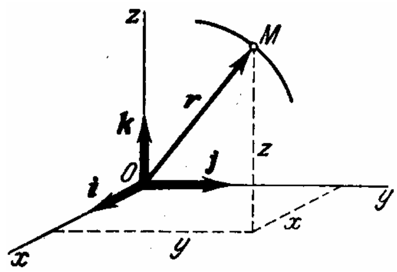


Fig. 6.1

Because the particle is in motion (changes its potion in time) the position vector is a function of time:

$$\mathbf{r} = \mathbf{r}(t) \qquad (6.1)$$

This function of time, for represents a real motion will meet the following conditions: it is continuous (the particle cannot make instantaneous jumps), it is uniformly (the particle cannot have more positions simultaneously) and it is derivable.

Coordinate method of describing motion. If we want to express the position of the particle in scalar way we know that, with respect to a reference system, for example the Cartesian reference system (Fig. 6.2), the position of the particle may be expressed using three coordinates (three scalar position parameters). These coordinates are functions of time also having the same conditions as the position vector:

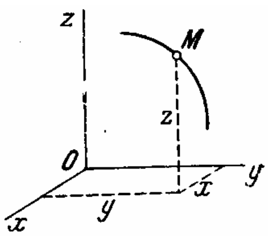


Fig. 6.2

$$x = x(t), \quad y = y(t), \quad z = z(t)\,. \qquad (6.2)$$

It is obviously that between the vector and the scalar expression of the potion we have the relation:

$$\mathbf{r}(t) = x(t)\mathbf{k} + y(t)\mathbf{j} + z(t)\mathbf{k}\,. \qquad (6.3)$$

Natural method of describing motion. The position of the particle can be expressed in another way also: we define the curved line (*C*) on which moves the particle and defines the position of the particle using the distance on this line with respect to a given position from the line (Fig. 6.3). The curved line on which the particle moves is called *trajectory* or *path* and by definition it is *the locus of the successively occupied positions of the particle in motion*. Noting that all positions from the trajectory can be defined using the position vector the trajectory may be defined also as *the locus of the position vector's peaks*.

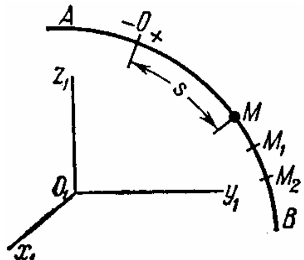


Fig. 6.3

If the parameter time has a given value, the position vector or the coordinates of the particle will be defined an *instantaneous position of the particle* (at a given instant). One of the important instantaneous position of the particle in the study of the motion is the *initial position*.

3. Velocity and acceleration

Let be a particle P in motion on an any trajectory. At the instant t of the motion the position of the particle will be defined by the position vector $\mathbf{r}(t)$. At another instant t_1:

$$t_1 = t + \Delta t$$

the position of the particle will be defined by the position vector:

$$\mathbf{r}_1 = \mathbf{r}(t_1) = \mathbf{r}(t + \Delta t) = \mathbf{r} + \Delta \mathbf{r}$$

where $\Delta \mathbf{r}$ is the variation of the position vector in the Δt interval of time (Fig. 6.4).

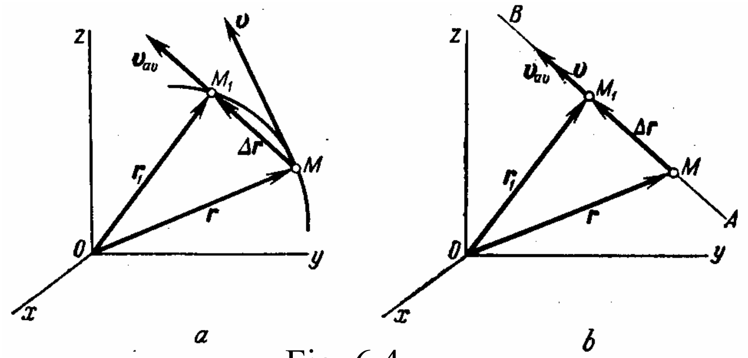


Fig. 6.4

We shall consider the following vector quantity defined by therelation:

$$\langle \boldsymbol{\upsilon} \rangle = \frac{\Delta \mathbf{r}}{\Delta t}$$

This vector is called *average velocity*. But we see that this vector does not correctly describe (than in particular cases) the kinds of motion. This rate, for example, if we consider a circular motion and the interval of time is equal to the time necessary to perform an entire circumference then the average velocity results equal to zero that is not true. Consequently this rate between the variation of the position vector and the corresponding interval of time is a feature of the motion only if the interval of time is very small (tends to zero). In this case we shall obtain the next vector:

$$\boldsymbol{\upsilon} = \lim_{\Delta t \to 0} \langle \boldsymbol{\upsilon} \rangle = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}. \tag{6.4}$$

This vector is called *instantaneous velocity* (at a given instant) and by definition is: *the first derivative with respect to time of the position vector*.

For to simplify we shall mark the first derivative with respect to time with a point above the derivate vector:

$$\boldsymbol{\upsilon} = \dot{\mathbf{r}}\,.$$

For to simplify the names in the problems we shall call the instantaneous velocity simply *velocity*. We shall use also the name *instantaneous velocity* but for the velocity at a given instant of the motion.

Consider now the particle in the two positions corresponding to the two instants: t and t_1. Because the velocities in these two positions are different, it is necessary, for to know the kind of motion of the particle to introduce a new notion that defines the variation of the velocity (Fig. 6.5). We shall bring the two velocities from the two instants in a convenient point. The variation of the velocity (as vector) in the interval of time is marked:

$$\langle \boldsymbol{a} \rangle = \frac{\Delta \boldsymbol{\upsilon}}{\Delta t}$$

that is called *average acceleration*. Because this vector does not describe well enough the kind of motion we shall define another notion decreasing the interval of time, finally obtaining the *instantaneous acceleration*:

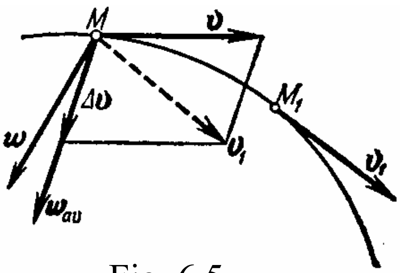


Fig. 6.5

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \langle \boldsymbol{a} \rangle = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{\upsilon}}{\Delta t} = \frac{d\boldsymbol{\upsilon}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \dot{\boldsymbol{\upsilon}} = \ddot{\mathbf{r}}\,. \tag{6.5}$$

Consequently the *instantaneous acceleration is the first derivative , with respect to time, of the velocity of the particle or the second derivative, with respect to time, of the position vector of the particle*.

As we can see the second derivative with respect to time is marked with two points above the corresponding vector.

4. Determination of the velocity and acceleration of a particle when its motion is described by the coordinate method

As we have seen in the previous sections the absolute motion of a particle can be studied using different reference systems. The simplest reference system is the *Cartesian system of reference* considered as a *fixed system*.

Consider a particle in motion (absolute motion) and a fixed Cartesian system of reference *Oxyz*.

The main property of this system can be expressed in the following way:

$$\frac{d\mathbf{i}}{dt} = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt} = 0 .$$

The position of the particle may be defined in scalar way using the three coordinates:

$$x = x(t), \quad y = y(t), \quad z = z(t) ,$$

that are functions of time because the particle is in motion (change its position) with respect to the fixed reference system.

These coordinates are called *the laws of motion in Cartesian coordinates or parametric equations of the motion in Cartesian coordinates*.

The position of the particle can be expressed also using the position vector with respect to the origin of the reference system:

$$\mathbf{r} = \mathbf{r}(t) .$$

Between this vector and the Cartesian coordinates we may write the well-known relation:

$$\mathbf{r}(t) = x(t)\mathbf{k} + y(t)\mathbf{j} + z(t)\mathbf{k} .$$

By eliminating time t from the equations of motion we can obtain the equation of the path in the usual form, i.e., in the form of a relation between the particle's coordinates.

For to know the kind of motion we shall express the velocity of the particle. Using the definition of the instantaneous velocity we find:

$$\boldsymbol{\upsilon}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} .$$

This means that the projections of the velocity on the axes of the reference system are:

$$\upsilon_x(t) = \frac{dx}{dt}, \quad \upsilon_y(t) = \frac{dy}{dt}, \quad \upsilon_z(t) = \frac{dz}{dt} ,$$

from which we obtain, using the well-known relations, the magnitude and the direction of the velocity in Cartesian coordinates:

$$\upsilon = \sqrt{\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2}, \quad \cos\alpha_\upsilon = \frac{\upsilon_x}{\upsilon}, \quad \cos\beta_\upsilon = \frac{\upsilon_y}{\upsilon}, \quad \cos\gamma_\upsilon = \frac{\upsilon_z}{\upsilon},$$

$$\cos^2\alpha_\upsilon + \cos^2\beta_\upsilon + \cos^2\gamma_\upsilon = 1.$$

We remark that the projections of the velocity on the fixed axes are equal to the first derivatives, with respect to time, of the corresponding coordinates.

Also we remark that in this reference system we have not any properties of the velocity resulted from the relations.

The second vector defining the kind of motion is the acceleration. From definition we have:

$$\boldsymbol{a} = \frac{d\boldsymbol{\upsilon}}{dt}$$

or removing function the Cartesian coordinates we obtain finally:

$$\boldsymbol{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

namely we have the following projections on the axes, magnitude and direction in Cartesian coordinates:

$$a_x(t) = \frac{d^2x}{dt^2}, \quad a_y(t) = \frac{d^2y}{dt^2}, \quad a_z(t) = \frac{d^2z}{dt^2},$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \quad \cos\alpha_a = \frac{a_x}{a}, \quad \cos\beta_a = \frac{a_y}{a}, \quad \cos\gamma_a = \frac{a_z}{a}.$$

5. Determination of the velocity and acceleration of a particle when its motion is described by the natural method. Tangential and normal accelerations of a particle

This reference system, called natural system also, is used only the cases when is known the trajectory of the particle.

Consider a particle P in motion on a known trajectory (*C*).

We shall consider the following reference system:

- The origin of the system is taken in the point representing the particle;

- The axis $P\tau$, called tangent axis, will be tangent to the trajectory in point P and with the positive sense in the sense of motion;

- The axis Pn, called normal axis, will have the direction of the principal normal to the trajectory in point P. The positive sense of this axis will be directed toward the center of curvature of the trajectory;

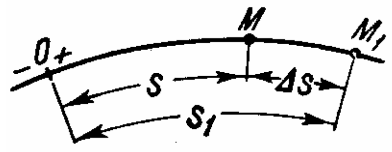


Fig. 6.6

- The axis Pb, called binormal axis, is perpendicular on the previous two axes and the positive sense is considered so that the three axes to make a right hand system.

Because the particle is located in the origin of this system and the names of the axes are not used to define coordinates, we shall use the names of these axes for the names of the corresponding unit vectors.

The position of the particle (because we know the trajectory of it) may be defined using one scalar quantity:

$$s = s(t),$$

called curvilinear coordinate or natural coordinate and representing the space performed on the trajectory measured from a convenient position (generally the initial position) to the current position. Because we study the absolute motion of the particle, for to define the velocity and acceleration we need to use the position vector with respect to a fixed point O.

For the velocity of the particle we have:

$$\upsilon_n = \upsilon_b = 0, \quad \upsilon_\tau = \pm\upsilon,$$

Let us see how the velocity of particle can be determined. If in a time interval $\Delta t = t_1$-t a particle moves from position M to position M_1 (Fig. 6.6), the displacement along the arc of the path being $\Delta s = s_1$-s, the numerical value of the average velocity will be:

$$\langle\upsilon\rangle = \frac{\Delta s}{\Delta t},$$

Passing to the limit, we obtain the numerical value of the instantaneous velocity for a given time t:

$$\upsilon = \lim_{\Delta t \to 0}\langle\upsilon\rangle = \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{6.6}$$

Thus, *the numerical value of the instantaneous velocity of a particle is equal to the first derivative of the dispacement (of the arc coordinate) s of the particle with respect to time.*

The velocity vector is *tangent* to the path, the latter assumed to be known.

Eq. (6.6) gives the *numerical* (*algebraic*) value of velocity, i.e., a quantity with a sign such that the sign of v is the same as the sign of Δs always $\Delta t > 0$. As the numerical value of the velocity vector differs from its magnitude only in sign, we shall denote both quantities by the same symbol v; this gives rise to practically no misunderstandings. Whenever it is necessary to stress that we are dealing with the magnitude of the velocity we shall denote it by the symbol $|\upsilon|$.

It was shown that the accelerations $\boldsymbol{a}$ of a particle lies in the osculating plane, i.e., plane $M\tau n$, hence the projection of vector $\boldsymbol{a}$ on the binormal is zero ($a_b = 0$).

Let us calculate the projections of $\boldsymbol{a}$ on the other two axes. Let the particle occupy a position M and have a velocity $\boldsymbol{\upsilon}$ at any time t, and at time $t_1 = t + \Delta t$ let it occupy a position M_1 and have a velocity $\boldsymbol{\upsilon}_1$. Then, by virtue of the definition,

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{\upsilon}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{\upsilon}_1 - \boldsymbol{\upsilon}}{\Delta t}.$$

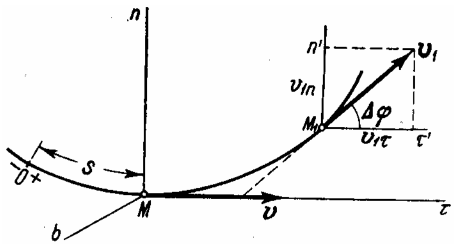


Fig. 6.7

Let us now express this equation in terms of the projections of the vectors on the axes $M\tau$ and Mn through point M (see Fig. 6.7). From the theorem of the projection of a vector sum (or difference) on an axis we obtain:

$$a_\tau = \lim_{\Delta t \to 0} \frac{\upsilon_{1\tau} - \upsilon_\tau}{\Delta t}, \quad a_n = \lim_{\Delta t \to 0} \frac{\upsilon_{1n} - \upsilon_n}{\Delta t}.$$

Noting that projections of a vector on parallel axes are equal, draw through point M_1 axes $M\tau'$ and Mn' parallel to $M\tau$ and Mn, respectively, and denote the angle between the direction of vector $\boldsymbol{\upsilon}_1$ and the tangent $M\tau$ by the symbol $\Delta\varphi$. This angle between the tangents to the curve at points M and M_1 is called the *angle of contiguity.*

It will be recalled that the limit of the ratio of the angle of contiguity $\Delta\varphi$ to the arc $MM_1 = \Delta s$ defines the curvature k of the curve at point M. As the curvature is the inverse of the radius of curvature ρ at M, we have:

$$\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta t} = k = \frac{1}{\rho}.$$

From the diagram in Fig. 6.7, we see that the projections of vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon}_1$ on the axes $M\tau$ and Mn are*

$$\upsilon_\tau = \upsilon, \qquad\qquad \upsilon_n = 0,$$
$$\upsilon_{1\tau} = \upsilon_1 \cos\Delta\varphi, \ \upsilon_{1\tau} = \upsilon_1 \sin\Delta\varphi,$$

where υ and υ_1 are the numerical values of the velocity of the particle at instants t and t_1. Hence,

$$a_\tau = \lim_{\Delta t\to 0}\frac{\upsilon_1\cos\Delta\varphi - \upsilon}{\Delta t}, \quad a_n = \lim_{\Delta t\to 0}\upsilon_1\frac{\sin\Delta\varphi}{\Delta t}.$$

It will be noted that when $\Delta t\to 0$, point M_1 approaches M indefinitely, and simultaneously $\Delta\varphi\to 0$, $\Delta s\to 0$, and $\upsilon_1\to\upsilon$.

Hence, taking into account that

$$\lim_{\Delta t\to 0}\cos\Delta\varphi = 1, \quad \lim_{\Delta t\to 0}\sin\Delta\varphi = \Delta\varphi .$$

we obtain for a_τ the expression

$$a_\tau = \lim_{\Delta t\to 0}\frac{\upsilon_1 - \upsilon}{\Delta t} = \frac{d\upsilon}{dt}.$$

We shall transform the right-hand side of the equation for w_n in such a way so that it includes ratios with known limits. For the purpose, multiplying the numerator and denominator of the fraction under the limit sign by $\Delta\varphi\Delta s$, we find:

$$a_n = \lim_{\Delta t\to 0}\upsilon_1\frac{\sin\Delta\varphi}{\Delta t} = \lim_{\Delta t\to 0}\upsilon_1\frac{\Delta\varphi}{\Delta s}\frac{\Delta s}{\Delta t} = \frac{\upsilon^2}{\rho}.$$

Finally we obtain

$$a_\tau = \frac{d\upsilon}{dt}, \quad a_n = \frac{\upsilon^2}{\rho}. \tag{6.6}$$

We have thus proved that *the projection of the acceleration of a particle on the tangent to the path is equal to the first derivative of the numerical value of the velocity, or the second derivative of the displacement (the arc coordinate) s, with respect to time; the projection of the acceleration on the principal normal is equal to the second power of the velocity divided by the radius of curvature of the path at the given point of the curve*, the projection of the acceleration on the binormal is zero (a_b=0). This is an important theorem of particle kinematics.

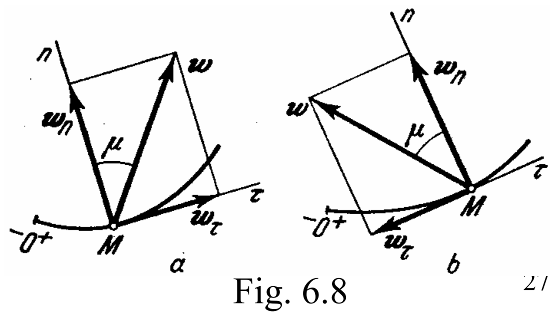


Fig. 6.8

Lay off vectors $\boldsymbol{a}_\tau$ and $\boldsymbol{a}_n$, i.e., the normal and tangential components of the acceleration, along the tangent $M\tau$ and the principal normal Mn, respectively (Fig. 6.8). The component $\boldsymbol{a}_n$ is always directed along the inward normal, as $a_n>0$, while the component $\boldsymbol{a}_\tau$ can be directed either in the positive or in the negative direction of the axis $M\tau$, depending on the sign of the projection $\boldsymbol{a}_\tau$ (see Figs. 6.8*a* and *b*).

The acceleration vector $\boldsymbol{a}$ is the diagonal of a parallelogram constructed with the components $\boldsymbol{a}_\tau$ and $\boldsymbol{a}_n$ as its sides. As the components are mutually perpendicular, the magnitude of vector $\boldsymbol{a}$ is given by the equation:

$$a = \sqrt{a_\tau^2 + a_n^2} = \sqrt{\left(\frac{d\upsilon}{dt}\right)^2 + \left(\frac{\upsilon^2}{\rho}\right)^2}. \qquad (6.7)$$

LECTURE 7
TRANSLATION AND ROTATIONAL MOTION OF A RIGID BODY
1. Translation motion

In kinematics, as in statics, we shall regard all solids as rigid bodies, i.e., we shall assume that the distance between any two points of a body remains the same during the whole period of motion.

Problems of kinematics of rigid bodies are basically of two types: (1) definition of the motion and analysis of the kinematic characteristics of the motion of a body as a whole; (2) analysis of the motion of every point of the body in particular.

We shall begin with the consideration of the motion of translation of a rigid body.

Translation of a rigid body is such a motion in which any straight line through the body remains continually parallel to itself. (Fig. 7.1)

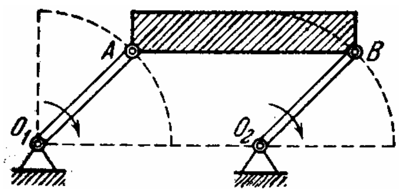


Fig. 7.1

The properties of translational motion are defined by the following theorem: *In translational motion, all the particles of a body move along similar paths* (*which will coincide if superimposed*) *and have at any instant the same velocity and acceleration.*

It follows from the theorem that the translational motion of a rigid body is fully described by the motion of any point

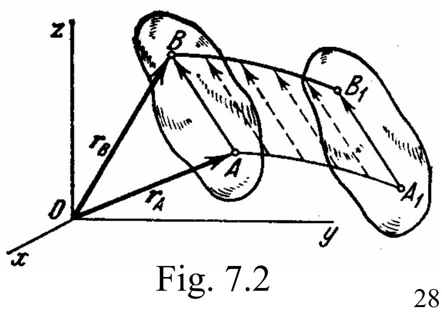


Fig. 7.2

belonging to it. Thus, the analysis of translational motion of a rigid body is reduced to the methods of particle kinematics examined before.

The common velocity $\boldsymbol{\upsilon}$ of all the points of a body in translational motion is called the *velocity of translation,* and the common acceleration $\boldsymbol{w}$ is called the *acceleration of translation.* Vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{a}$ can, obviously, be shown as applied at any point of the body.

2. Rotational motion of a rigid body. Angular velocity and angular acceleration

Rotation of a rigid body is such a motion in which there are always two points of the body (*or body extended*) *which remain motionless* (see Fig. 7.3). The line *AB* through these fixed points is called the *axis of rotation*.

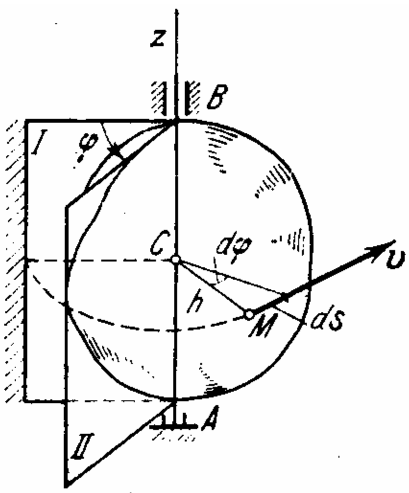


Fig. 7.3

To determine the position of a rotating body, let us pass two planes through the axis of rotation *Az:* plane *I,* which is fixed, and plane *II* through the rotating body and rotating with it (Fig. 7.3). The position of the body at any instant will be fully specified by the angle φ between the two planes, taken with the appropriate sign, which we shall call the *angle of rotation* of the body. We shall consider the angle positive if it is laid off counterclockwise from the fixed plane by an observer looking from the positive end of axis *Az,* and negative if it is laid off clockwise. Angle φ is always measured in *radians.*

The position of a body at any instant is completely specified if we know the angle φ as a function of time *t,* i.e.,

$$\varphi = \varphi(t)\,. \tag{7.1}$$

Eq. (7.1) describes the *rotational motion of a rigid body.*

The principal kinematic characteristics of the rotation of a rigid body are its *angular velocity* ω and *angular acceleration* ε.

The angular velocity of a body at a given time is equal in magnitude to the first derivative of the angle of rotation with respect to time.

$$\omega = \frac{d\varphi}{dt}\,. \tag{7.2}$$

Eq. (7.2) also shows that the value of ω is equal to the ratio of the infinitesimal angle of rotation $d\varphi$ to the corresponding time interval dt. The sign of φ specifies the direction of the rotation. It will be noticed that $\omega>0$ when the rotation is counterclockwise, and $\omega<0$ when the rotation is clockwise. The dimension of angular velocity, if the time is measured in seconds, is

$$[\omega]=\frac{\text{radian}}{\text{sec}}=\text{s}^{-1}.$$

as the radian is a dimensionless unit.

The angular velocity of a body can be denoted by a vector **ω** of magnitude ω along the axis of rotation of the body in the direction from which the rotation is seen as counterclockwise (see Fig. 7.4). Such a vector simultaneously gives the magnitude of the angular velocity, the axis of rotation, and the sense of rotation about that axis.

The angular acceleration of a body at a given time is equal in magnitude to the first derivative of the angular velocity, or the second derivative of the angular displacement, of the body with respect to time.

$$\varepsilon=\frac{d\omega}{dt}. \qquad (7.3)$$

The dimension of angular acceleration is $[\varepsilon] = \text{s}^{-2}$.

If the angular velocity increases in magnitude, the rotation is *accelerated*, if it decreases, the rotation is *retarded.* It will be readily noticed that the rotation is accelerated when ω and ε are of the same sign, and retarded when they are of different sign.

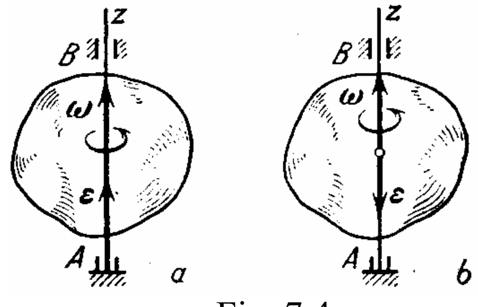


Fig. 7.4

By analogy with angular velocity, the angular acceleration of a body can be denoted by a vector **ε** along the axis of rotation. The direction of **ε** coincides with that of **ω** when the rotation is accelerated (Fig. 7.4*a*), and is of opposite sense when the rotation is retarded (Fig. 7.4*b*).

3. Velocities and accelerations of the points of a rotating body

Consider a point M of a rigid body at a distance h from the axis of rotation Az (Fig. 7.3). When the body rotates, point M describes a circle of radius h in a plane perpendicular to the axis of rotation with its centre C on that axis. If in time dt the body makes an infinitesimal displacement through an angle $d\varphi$, point M will have made a very small displacement $ds = h\,d\varphi$ along its path. The velocity of the point is the ratio of ds to dt, i.e.,

$$\upsilon = \frac{ds}{dt} = h\frac{d\varphi}{dt},$$

or

$$\upsilon = h\omega. \qquad (7.4)$$

This velocity υ is called the *linear*, or *circular*, velocity of the point M (not to be confused with its angular velocity).

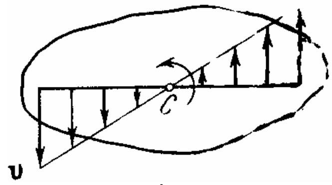


Fig. 7.5

Thus, *the linear velocity of a point belonging to a rotating body is equal to the product of the angular velocity of that body and the distance of the point from the axis of rotation.* The linear velocity is tangent to the circle described by point M, or perpendicular to the plane through the axis of rotation and the point M.

In order to determine the acceleration of point M, we apply equations

$$a_\tau = \frac{d\upsilon}{dt}, \quad a_n = \frac{\upsilon^2}{\rho}.$$

In our case, $\rho = h$. Substituting the expression for υ from Eq. (7.4), we obtain

$$a_\tau = h\frac{d\omega}{dt}, \quad a_n = \frac{h^2\omega^2}{h},$$

and finally

$$a_\tau = h\varepsilon, \quad a_n = h\omega^2. \qquad (7.5)$$

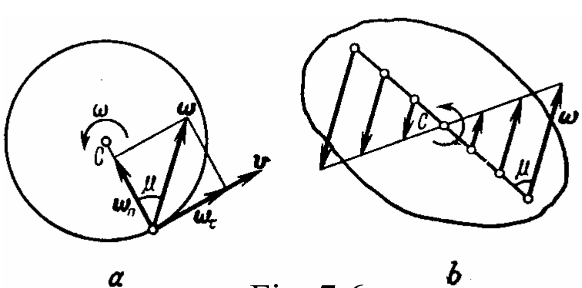


Fig. 7.6

The tangential acceleration a_τ is tangent to the path (in the direction of the rotation if it is accelerated and in the reverse direction if it is retarded); the normal acceleration a_n is always directed along the radius h towards the axis of rotation (Fig. 7.6*a*).

The total of point M is

$$a = \sqrt{a_\tau^2 + a_n^2} = \sqrt{h^2\varepsilon^2 + h^2\omega^4},$$

or

$$a = h\sqrt{\varepsilon^2 + \omega^4}\,. \tag{7.6}$$

LECTURE 8
PLANE MOTION OF A RIGID BODY
1. Equations of plane motion. Resolution of motion into translation and rotation

Plane motion of a rigid body is such motion in which all its points move parallel to a fixed plane P (Fig. 8.1). Many machine parts have plane motion, for example, a wheel running on a straight track or the connecting rod of a reciprocating engine. Rotation is, in fact, a special case of plane motion.

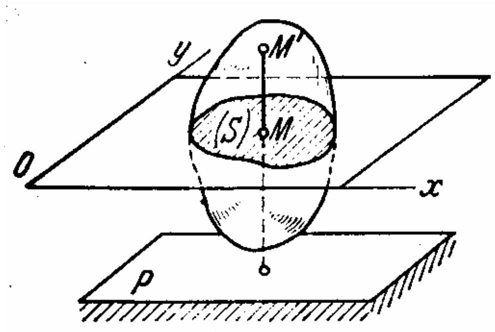


Fig. 8.1

Let as consider the section *S* of a body produced by passing any plane *Oxy* parallel to a fixed plane *P* (see Fig. 8.1). All the points of the body belonging to line *MM'* normal to plane *P* move in the same way. Therefore, *in investigating plane motion it is sufficient to investigate the motion of section S of that body* in the plane *Oxy.* In this book we shall always take the plane *Oxy* parallel to the page and represent a body by its section *S*.

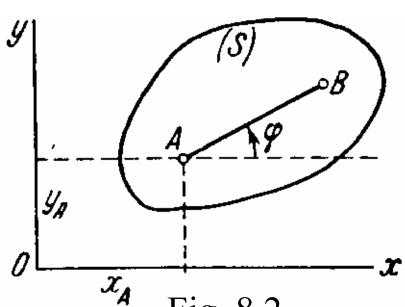


Fig. 8.2

The position of section *S* in plane *Oxy* is completely specified by the position of any line *AB* in this section (Fig. 8.2). The position of the line *AB* may be specified by the coordinates x_A and y_A of point *A* and the angle φ between an arbitrary line *AB* in section *S* and axis *x*.

The point *A* chosen to define the position of section *S* is called the *pole.* As the body moves, the quantities x_A, y_A and φ will change and the motion of the body, i.e., its position in space at any moment of time, will be completely specified if we know.

$$x_A = x(t), \quad y_A = y(t), \quad \varphi = \varphi(t). \tag{8.1}$$

Eqs.(8.1) are the *equations of plane motion of a rigid body.*

We conclude that *the plane motion of a rigid body is a combination of a translation, in which all the points move in the same way as the pole A, and of a rotation about that pole* *.

The *principal kinematic characteristics* of this type of motion are the velocity and acceleration of translation, each equal to the velocity and acceleration of the pole ($\boldsymbol{v}_{\text{trans}} = \boldsymbol{v}_A$, $\boldsymbol{a}_{\text{trans}} = \boldsymbol{a}_A$), and the angular velocity ω and angular acceleration ε of the rotation about the pole. The values of these characteristics can be found for any time t from Eqs. (8.1).

In analysing plane motion, we are free to choose any point of the body as the pole. Let us consider a point C as a pole instead of A and determine the position of the line CD making an angle φ_1 with axis x (Fig. 8.3). The characteristics of the translatory component of the motion would have been different, for in the general case $\boldsymbol{v}_C \neq \boldsymbol{v}_A$ and $\boldsymbol{a}_C \neq \boldsymbol{a}_A$ (otherwise the motion would be that of pure translation). The characteristics of the rotational component of the motion ω and ε remain, however, the same. For, drawing CB_1 parallel to AB, we find that at any instant of time angle $\varphi_1 = \varphi - \alpha$, where $\alpha = \text{const}$. Hence

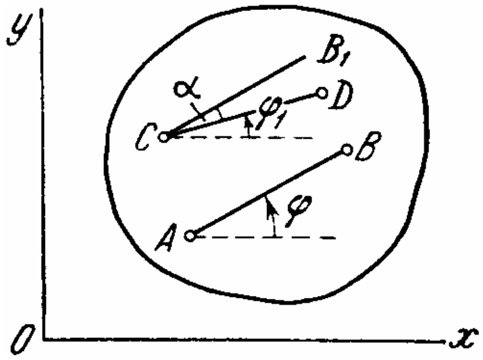


Fig. 8.3

$$\frac{d\varphi_1}{dt} = \frac{d\varphi}{dt}, \quad \frac{d^2\varphi_1}{dt^2} = \frac{d^2\varphi}{dt^2},$$

or

$$\omega_1 = \omega, \quad \varepsilon_1 = \varepsilon.$$

Hence, *the rotational component of motion does not depend on the position of the pole.*

2. Determination of the velocity of a point of a body

Plane motion of a rigid body is a combination of a translation in which all points of the body move with the velocity of the pole $\boldsymbol{v}_A$ and a rotation about that pole. Let us show that the velocity of any point M of the body is the geometrical sum of its velocities for each component of the motion.

The position of a point M in section S of the body is specified with

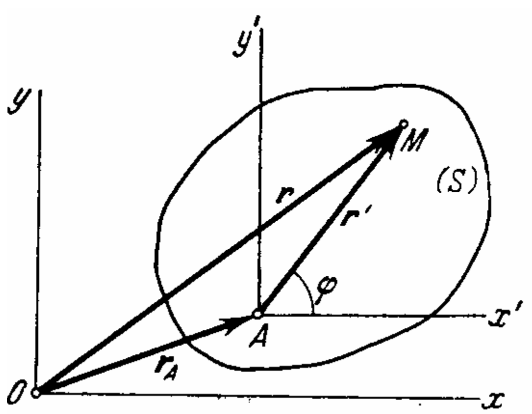


Fig. 8.4

reference to the coordinate axes *Oxy* by the radius vector $\mathbf{r} = \mathbf{r}_A + \mathbf{r}'$ (Fig. 8.4), where $\mathbf{r}_A$ is the radius vector of the pole *A*, $\mathbf{r}'$=**AM** is the vector which specifies the position of point *M* with reference to the axes *Ax'y'* that perform translational motion together with *A* (the motion of section *S* with reference to those axes is the motion about pole *A*). Then,

$$\boldsymbol{\upsilon}_M = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}'}{dt}.$$

In this equation first term is equal to the velocity of pole *A*; the second term is equal to the velocity $\boldsymbol{\upsilon}_{mA}$ of point *M* at $\mathbf{r}_A$ = const., i.e., when *A* is fixed or, in other words, *when the body* (or, strictly speaking, its section *S*) *rotates about pole A*. It thus follows from the preceding equation that

$$\boldsymbol{\upsilon}_M = \boldsymbol{\upsilon}_A + \boldsymbol{\upsilon}_{MA}. \qquad (8.2)$$

The velocity of rotation $\boldsymbol{\upsilon}_{mA}$ of point *M* about pole *A* is

$$\upsilon_{MA} = \omega MA \quad (\boldsymbol{\upsilon}_{MA} \perp \mathbf{MA}), \qquad (8.3)$$

where ω is the angular velocity of the rotation of the body.

Thus, the velocity of any point *M* of a body is the geometrical sum of the velocity of any other point *A* taken as the pole and the velocity of rotation of point *M* about the pole. The magnitude and direction of the velocity $\boldsymbol{\upsilon}_m$ are found by constructing a parallelogram (Fig. 8.5).

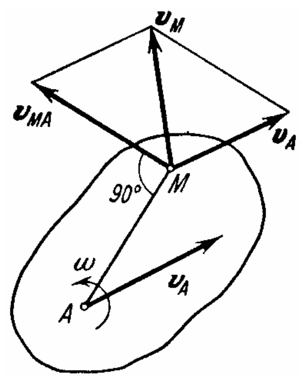


Fig. 8.5

3. Theorem of the projections of the velocities of two points of a body

The use of Eq. (8.2) to determine the velocities of the points of a body usually leads to involved computations. However, we can evolve from Eq. (8.2) several simpler and more convenient methods of determining the velocity of any point of a body.

One of these methods is given by the theorem: *The projections of the velocities of two points of a rigid body on the straight line joining those points are equal.*

Consider any two points *A* and *B* of a body. Taking point *A* as the pole (Fig. 8.6) we have from Eq. (50) $\boldsymbol{\upsilon}_B = \boldsymbol{\upsilon}_A + \boldsymbol{\upsilon}_{BA}$. Projecting both members of the equation on

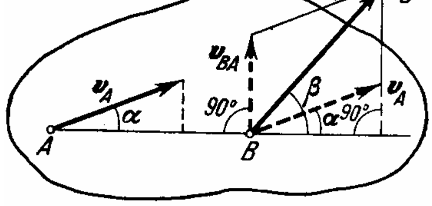


Fig. 8.6

AB and taking into account that vector $\boldsymbol{\upsilon}_{BA}$. is perpendicular to AB, we obtain:

$$\upsilon_A \cos\alpha = \upsilon_B \cos\beta\,. \tag{8.4}$$

and the theorem is proved. This result offers a simple method of determining the velocity of any point of a body if the direction of motion of that point and the velocity of any other point of the same body are known.

LECTURE 9
PLANE MOTION OF A RIGID BODY (continuation)
4. Determination of the velocity of a point of a body using the instantaneous centre of zero velocity

Another simple and visual method of determining the velocity of any point of a body performing plane motion is based on the concept of instantaneous centre of zero velocity. *The instantaneous centre of zero velocity is a point belonging to the section S of a body or its extension which at the given instant is momentarily at rest*.

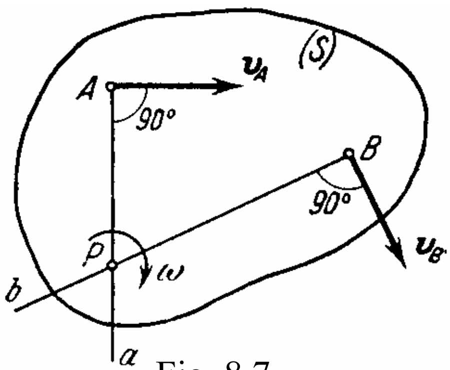


Fig. 8.7

It will be readily noticed that if a body is in non-translational motion, such one and only one point always exists at any instant *t*. Let points A and B in section S of a body (Fig. 8.7) have, at time *t*, non-parallel velocities $\boldsymbol{\upsilon}_A$ and $\boldsymbol{\upsilon}_B$. Then point P of intersection of perpendiculars Aa to vector $\boldsymbol{\upsilon}_A$ and Bb to vector $\boldsymbol{\upsilon}_B$ will be the instantaneous centre of zero velocity, as $\boldsymbol{\upsilon}_P = 0$. For, if we assumed that $\boldsymbol{\upsilon}_P \neq 0$, then, by the theorem of the projections of the velocities of the points of a body, vector $\boldsymbol{\upsilon}_P$ would have to be simultaneously perpendicular to AP (as $\boldsymbol{\upsilon}_A \perp AP$) and to BP (as $\boldsymbol{\upsilon}_B \perp BP$), which is impossible. It also follows from the theorem that, at the given instant, no other point of section S can have zero velocity (e.g., for point *a*, the projection of $\boldsymbol{\upsilon}_B$ on Ba is not zero and consequently $\boldsymbol{\upsilon}_a \neq 0$).

If, now, we take a point P as the pole at time *t*, the velocity of point A will, by Eq. (8.2), be

$$\boldsymbol{\upsilon}_A = \boldsymbol{\upsilon}_P + \boldsymbol{\upsilon}_{AP} = \boldsymbol{\upsilon}_{AP}\,,$$

as $\boldsymbol{\upsilon}_P = 0$. The same result can be obtained for any other point of the body. Thus, *the velocity of any point of a body lying in section S is equal to the* *velocity of its rotation about the instantaneous centre of zero velocity P*. From Eqs. (8.2) we have

$$\upsilon_A = \omega PA \quad (\boldsymbol{\upsilon}_A \perp \mathbf{PA}),$$
$$\upsilon_B = \omega PB \quad (\boldsymbol{\upsilon}_B \perp \mathbf{PB}), \quad \text{etc.}, \tag{8.5}$$

It also follows from Eqs. (8.5) that

$$\frac{\upsilon_A}{PA} = \frac{\upsilon_B}{PB}, \tag{8.6}$$

i.e., that *the velocity of any point of a body is proportional to its distance from the instantaneous centre of zero velocity*.

These results lead to the following conclusions:

(1) *To determine the instantaneous centre of zero velocity, it is sufficient to know the directions of the velocities* $\boldsymbol{\upsilon}_A$ *and* $\boldsymbol{\upsilon}_B$ *of any two points A and B of a section of a body* (or their paths); the instantaneous centre of zero velocity lies at the intersection of the perpendiculars erected from points *A* and *B* to their respective velocities, or to the tangents to their paths.

(2) *To determine the velocity of any point of a body, it is necessary to know the magnitude and direction of the velocity of any point A of that body and the direction of the velocity of another point B of the same body.* Then, by erecting from points *A* and *B* perpendiculars to $\boldsymbol{\upsilon}_A$ and $\boldsymbol{\upsilon}_B$, we obtain the instantaneous centre of zero velocity *P* and, from the direction of $\boldsymbol{\upsilon}_A$, the sense of rotation of the body. Next, knowing $\boldsymbol{\upsilon}_A$, we can find from Eq. (54) the velocity $\boldsymbol{\upsilon}_M$ of any point *M* of the body. Vector $\boldsymbol{\upsilon}_M$ is perpendicular to **PM** in the direction of the rotation.

(3) *The angular velocity of a body*, as can be seen from Eqs. (8.5), *is at any given instant equal to the ratio of the velocity of any point belonging to the section S to its distance from the instantaneous centre of zero velocity P*:

$$\omega = \frac{\upsilon_B}{PB}, \tag{8.7}$$

Let us evolve another expression for ω It follows from Eqs. (8.2) and (8.3) that $\boldsymbol{\upsilon}_{BA} = \left| \boldsymbol{\upsilon}_B - \boldsymbol{\upsilon}_A \right|$ and $\upsilon_{BA} = \omega AB$ whence

$$\omega = \frac{|\boldsymbol{\upsilon}_B - \boldsymbol{\upsilon}_A|}{AB} = \frac{|\boldsymbol{\upsilon}_B + (-\boldsymbol{\upsilon}_A)|}{AB}. \tag{8.8}$$

When $\upsilon_A = 0$ (point A is the instantaneous centre of zero velocity), Eq. (8.8) transforms into Eq. (8.7).

Eqs. (8.7) and (56) give the same quantity, since the rotation of the section S about either point A or point P takes place with the same angular velocity ω.

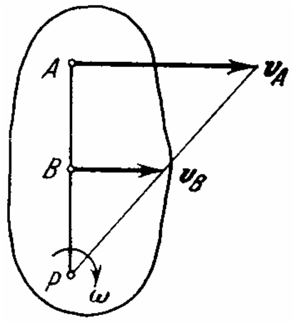


Fig. 8.10

Let us consider some special cases of the instantaneous centre of zero velocity.

(a) If plane motion is performed by a cylinder rolling without slipping along a fixed cylindrical surface, the point of contact P (Fig. 8.8) is momentarily at rest and, consequently, is the instantaneous centre of zero velocity ($\boldsymbol{\upsilon}_P = 0$ because if there is no slipping, the contacting points of both bodies must have the same velocity, and the second body is motionless). An example of such motion is that of a wheel running on a rail.

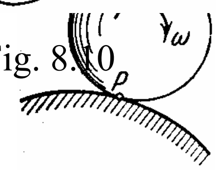


Fig. 8.8

(b) If the velocities of points A and B of the body are parallel to each other, and AB is not perpendicular to $\boldsymbol{\upsilon}_A$ (Fig. 8.9) the instantaneous centre of zero velocity lies in infinity, and the velocities of all points are parallel to $\boldsymbol{\upsilon}_A$. From the theorem of the projections of velocities it follows that $\upsilon_A\cos\alpha = \upsilon_B\cos\beta$, i.e., $\upsilon_B = \upsilon_A$; the result is the same for all other points of the body. Consequently, in this case the velocities of all points of the body are equal in magnitude and direction at every instant, i.e., *the instantaneous distribution of the velocities of the body is that of translation* (this state of motion is also called *instantaneous translation*). It will be found from Eq. (8.8) that the angular velocity ω of the body at the given instant is zero.

(c) If the velocities of points A and B are parallel and is perpendicular to $\boldsymbol{\upsilon}_A$, the instantaneous centre of zero velocity P can be located by the construction shown in Fig. 8.10. The validity of this construction follows from the proportion (8.6). In this case, unlike the previous ones, we have to know the magnitudes of velocities υ_A and υ_B to locate the instantaneous centre of zero velocity P.

(d) If the velocity vector $\boldsymbol{\upsilon}_B$ of a point in section S and the angular velocity ω are known, the position of the instantaneous centre of zero velocity P, lying on the perpendicular to $\boldsymbol{\upsilon}_B$ (see Fig. 8.7), can be immediately found from Eq. (8.7), which yields $BP = \upsilon_B/\omega$.

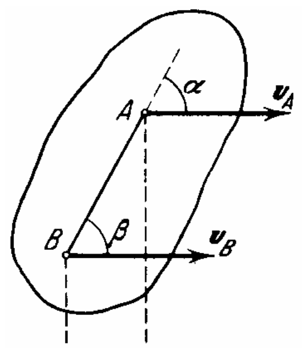


Fig. 8.9

5. Determination of the acceleration of a point of a body

We shall demonstrate that, like velocity, the acceleration of any point M of a body in plane motion is composed of its accelerations of translation and rotation. The location of point M with respect to axes Oxy (see Fig. 8.4) is specified by the radius vector $\mathbf{r} = \mathbf{r}_A + \mathbf{r}'$, where $\mathbf{r}' = \mathbf{AM}$. Hence,

$$\boldsymbol{a}_M = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}_A}{dt^2} + \frac{d^2\mathbf{r}'}{dt^2}.$$

In this equation the quantity first term is the acceleration of the pole A, and the second term is the acceleration of point M in its rotation with the body round A. Hence,

$$\boldsymbol{a}_M = \boldsymbol{a}_A + \boldsymbol{a}_{MA}. \qquad (8.9)$$

From Eqs. (7.5) and (7.6), the acceleration of point M in its rotation about A is

$$a_{MA} = MA\sqrt{\omega^4 + \varepsilon^2}. \qquad (8.10)$$

where ω and ε are the angular velocity and angular acceleration of the body.

Thus, *the acceleration of any point M of a body is composed of the acceleration of any other point taken for the pole and the acceleration of the point M in its rotation together with the body about that pole*. The magnitude and direction of the acceleration $\boldsymbol{a}_M$ are determined by constructing a parallelogram (Fig. 8.11).

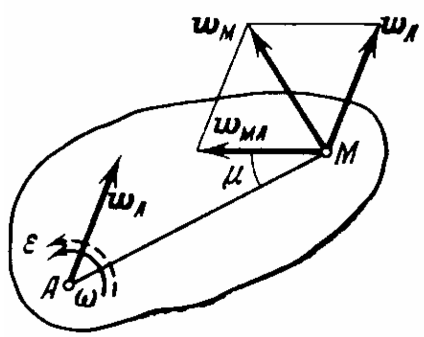


Fig. 8.11

However, the computation of $\boldsymbol{a}_M$ by means of the parallelogram in Fig. 8.11 makes the solution more difficult, as it becomes necessary first to calculate the angle and then the angle between vectors $\boldsymbol{a}_{MA}$ and $\boldsymbol{a}_A$ Therefore, in problem solutions it is more convenient to replace vector $\boldsymbol{a}_{MA}$ by its tangental and normal components $\boldsymbol{a}^\tau_{MA}$ and $\boldsymbol{a}^n_{MA}$, where

$$a^\tau_{MA} = MA\varepsilon, \quad a^n_{MA} = MA\omega^2. \qquad (8.11)$$

Vector $\boldsymbol{a}^{\tau}_{MA}$ is perpendicular to AM in the direction of the rotation if it is accelerated, and opposite the rotation if it is retarded; vector $\boldsymbol{a}^{n}_{MA}$ is always directed from point M to the pole A (Fig. 8.12).

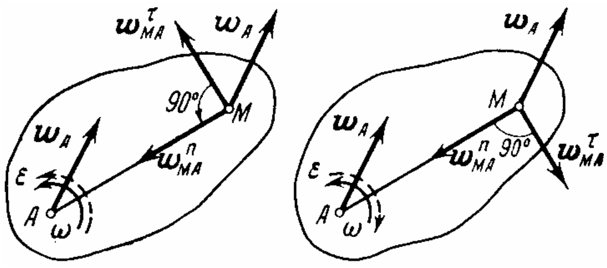


Fig. 8.12

Instead of Eq. (8.9) we obtain

$$\boldsymbol{a}_M = \boldsymbol{a}_A + \boldsymbol{a}^{\tau}_{MA} + \boldsymbol{a}^{n}_{MA}. \tag{8.12}$$

If pole A is in non-rectilinear motion, its acceleration is also composed of the tangential and normal accelerations, hence

$$\boldsymbol{a}_M = \boldsymbol{a}^{\tau}_{A} + \boldsymbol{a}^{n}_{A} + \boldsymbol{a}^{\tau}_{MA} + \boldsymbol{a}^{n}_{MA}. \tag{8.13}$$

the magnitudes of the latter two components being obtained from Eq. (8.11). Eqs. (8.11), (8.12) and (8.13) should be used in solving problems, first computing the vectors in the right-hand part of the equation and then finding their geometrical sum or making a graphic construction.

LECTURE 10
PARTICLE DYNAMICS

Dynamics is the section of mechanics which treats of the laws of motion of material bodies subjected to the action of forces.

The motion of bodies from a purely geometrical point of view was discussed in kinematics. Unlike kinematics, in dynamics the motion of bodies is investigated in connection with the acting forces and the inertia of the material bodies themselves.

The concept of force as a quantity characterising the measure of mechanical interaction of material bodies was introduced in the course of statics. But in statics we treated all forces as constant, without considering the possibility of their changing with time. In real systems, alongside of constant forces (gravity can generally be regarded as an example of a constant force), a body is often subjected to the action of variable forces whose magnitudes and directions change when the body moves. Variable forces may be the applied (active) forces or the reactions of constraints.

Experience shows that variable forces may depend in some specific ways on *time*, on the *position* of a body, and on its *velocity* (examples of dependence on *time* are furnished by the tractive force of an electric locomotive whose rheostat is gradually switched on or off or the force causing the vibration of the foundation of a motor with a poorly centred shaft; the Newtonian force of gravitation or the elastic force of a spring depending on the *position* of a body; the resistance experienced by a body moving through air or water depends on the *velocity*. In dynamics we shall deal with such forces alongside of constant forces. The laws for the composition and resolution of variable forces are the same as for constant forces.

The concept of inertia of bodies arises when we compare the results of the action of an identical force on different material bodies. Experience shows that if the same force is applied to two different bodies initially at rest and free from any other actions, in the most general case the bodies will travel different distances and acquire different velocities in the same interval of time.

Inertia is the property of material bodies to resist a change in their velocity under the action of applied forces. If, for example, the velocity of one body changes slower than that of another body subjected to the same force, the former is said to have greater inertia, and vice versa.

The quantitative measure of the inertia of a body is a physical quantity called the mass of that body. In mechanics mass m is treated as a quantity which is positive and constant for every body.

In the most general case the motion of a body depends not only on its total mass and the applied forces; the nature of motion may also depend on the shape of the body or, more precisely, on the mutual position of its particles (i.e., on the distribution of mass).

In the initial course of dynamics, so as to neglect the influence of the shape (distribution of the mass) of a body, the concept of a *material point,* or *particle* is introduced.

A particle is a material body (a body possessing mass) the size of which can be neglected in investigating its motion.

Actually any body can be treated as a particle when the distances travelled by its points are very great as compared with the size of the body itself. For example, in studying the motion of a planet about the sun or determining the range of an artillery shell, etc., the planet or shell can be treated as particles. Furthermore, as will be shown in the dynamics of systems, a body in *translational* motion can always be considered as a particle of mass equal to the mass of the whole body. Finally, the parts into which we shall mentally divide bodies in analysing any of their dynamical characteristics can also be treated as material points.

Obviously, the investigation of the motion of a single particle should precede the investigation of systems of particles, and in particular of rigid bodies. Accordingly, the course of dynamics is conventionally subdivided into particle dynamics and the dynamics of systems of particles.

1. The laws of dynamics

The study of dynamics is based on a number of laws generalising the results of a wide range of experiments and observations of the motions of bodies–laws which have been verified in the long course of human history. These laws were first systematised and formulated by Isaac IN ewton in his classical work *Principia Mathematica* published in 1687.

The **First Law** (the Inertia Law), discovered by Galileo in 1638, states: *A particle free from any external influences continues its state of rest or uniform rectilinear motion, except and so far as it is compelled to change that state by impressed forces*. The motion of a body not subjected to any force is called motion under no forces, or *inertial motion*.

The inertia law states one of the basic properties of matter: that of being always in motion. It establishes the equivalence, for material bodies, of the states of rest and of motion under no forces. It follows, then, that if $\mathbf{F} = 0$, a particle is at rest or moves with a velocity of constant magnitude and direction ($\boldsymbol{\upsilon}$ = const.); the acceleration of the particle is, evidently, zero ($\boldsymbol{a} = 0$); if the motion of a particle is not uniform and rectilinear, there must be some force acting on it.

A frame of reference for which the inertia law is valid is called an *inertial frame* (or, conventionally, a fixed frame). Experience shows that, for our solar system, an inertial frame of reference has its origin in the centre of the sun and its axes pointed towards the so-called "fixed" stars. In solving most engineering problems a sufficient degree of accuracy is obtained by assuming any frame of reference connected with the earth to be an inertial system.

The **Second Law** (the Fundamental Law of Dynamics) establishes the mode in which the velocity of a particle changes under the action of a force: *The product of the mass of a particle and the acceleration imparted to it by a force is proportional to the acting force*; *the acceleration takes place in the direction of the force*.

Mathematically this law is expressed by the vector equation

$$m\boldsymbol{a} = \mathbf{F}\,. \qquad (10.1)$$

The dependence between the magnitudes of the acceleration and the force is

$$ma = F . \tag{10.2}$$

The second law of dynamics, like the first, is valid only for an inertial frame. It can be immediately seen from the law that the measure of the inertia of a particle is its mass, since two different particles subjected to the action of the same force receive the same acceleration only if their masses are equal; if their masses are different, the particle with the larger mass (i.e., the more inert one) will receive a smaller acceleration, and vice versa.

A set of forces acting on a particle can, as we know, be replaced by a single resultant $\boldsymbol{R}$ equal to the geometrical sum of those forces. In this case the equation expressing the fundamental law of dynamics acquires the form

$$m\boldsymbol{a} = \sum_k \mathbf{F}_k . \tag{10.3}$$

This result can also be obtained by applying, instead of the parallelogram principle, the *law of independent action of forces,* according to which each of a number of forces acting on a particle imparts to it the same acceleration as it would have imparted if acting alone.

Weight and Mass. All bodies close to the surface of the earth are subject to the force of gravity **P**, equal in magnitude to a body's weight. It has been established experimentally that under the action of force **P** all bodies falling to the earth (from a small height and in vacuo) possess the same acceleration g; this is known as the *acceleration of gravity* or *of free fall.* Applying Eq. (10.2), for free fall we have

$$mg = P \quad or \quad m = \frac{P}{g} . \tag{10.4}$$

Eq. (10.4) gives the body's mass in terms of its weight, and vice versa; it establishes that *a body's weight is the product of its mass and acceleration of gravity, and its mass is the quotient of its weight divided by the acceleration of gravity.* Weight, like the acceleration of gravity g, changes with latitude and altitude; mass is a constant for every given body (or particle).

The **Third Law** (the Law of Action and Reaction) establishes the character of mechanical interaction between material bodies. For two particles it states: *Two particles exert on each other forces equal in magnitude and acting in opposite directions along the straight line connecting the two particles.*

It should be noted that the forces of interaction between free particles (or bodies) do not form a balanced system, as they act on different objects.

For example, if a piece of iron and a magnet are placed near each other on a smooth surface, they will move towards each other under the influence of their mutual attraction and not remain at rest. Since the magnitude of the force acting on each body is the same, it follows from the second law of dynamics that the accelerations of the two bodies will be inversely proportional to their masses.

The third law of dynamics, which establishes the character of interaction of material particles, plays an important part in the dynamics of systems.

3. The problems of dynamics for a free and a constrained particle

The problems of dynamics for a *free* particle are: (1) knowing the equation of motion of a particle, determine the force acting on it (*the first problem of dynamics*); (2) knowing the forces acting on a particle, determine its equation of motion (*the second*, or *principal, problem of dynamics*).

Both problems are solved with the help of Eq. (1) or (3), which express the fundamental law of dynamics, since they give the relation between acceleration $\boldsymbol{a}$, i.e., the quantity characterising the motion of a particle, and the forces acting on it.

In engineering it is often necessary to investigate the *constrained* motion of a particle, i.e., cases when constraints attached to a particle compel it to move along a given fixed surface or curve.

In such cases we shall use, as in statics, the axiom of constraints which states that *any constrained particle can be treated as a free body detached from its constraints provided the latter is represented by their reactions* **N**. Then the *fundamental law of dynamics for the constrained motion of a particle* takes the form

$$m\boldsymbol{a} = \sum_k \mathbf{F}_k^{\alpha} + \mathbf{N}, \qquad (10.5)$$

where $\mathbf{F}^{\alpha}{}_k$ denotes the applied forces acting on the particle.

For constrained motion, the first problem of dynamics will usually be: determine the reactions of the constraints acting on a particle if the motion and applied forces are known. The second (principal) problem of dynamics for such motion will pose two questions, namely, knowing the applied forces, to determine: (a) the equation of motion of the particle and (b) the reaction of its constraints.

4. Rectilinear motion of a particle

We know from kinematics that in rectilinear motion the velocity and acceleration of a particle are continuously directed along the same straight line. As the direction of acceleration is coincident with the direction of force, it follows that a free particle will move in a straight line whenever the force acting on it is of constant direction and the velocity at the initial moment is either zero or is collinear with the force.

Consider a particle moving rectilinearly under the action of an applied force **R**. The position of the particle on its path is specified by its coordinate x (Fig. 10.1) In this case the principal problem of dynamics is: knowing **R**, find the equation of motion of the particle $x = f(t)$. Eq. (10.3) gives the relation between x and R. Projecting both sides of the equation on axis Ox, we obtain:

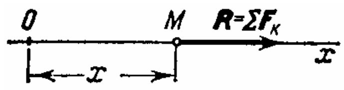


Fig. 10.1

$$ma_x = R_x = \sum_k F_{kx},$$

or, as

$$m\frac{d^2x}{dt^2} = \sum_k F_{kx}. \qquad (10.6)$$

Eq. (10.6) is called the *differential equation of rectilinear motion of a particle*. It is often more convenient to replace Eq. (10.6) with two differential equations containing first derivatives:

$$m\frac{d\upsilon_x}{dt} = \sum_k F_{kx}, \quad \frac{dx}{dt} = \upsilon_x. \qquad (10.7)$$

Whenever the solution of a problem requires that the velocity be found as a function of the coordinate x instead of time t (or when the forces themselves depend on x), Eq. (10.7) is converted to the variable x. Eq. (10.7) takes the form

$$m\upsilon_x\frac{d\upsilon_x}{dx} = \sum_k F_{kx}. \qquad (10.8)$$

The principal problem of dynamics is, essentially, to develop the equation of motion $x = f(t)$ for a particle from the above equations, the forces being known. For this it is necessary to integrate the corresponding differential equation. In order to make clear the nature of the mathematical problem, it should be recalled that the forces in the right side of Eq. (10.6) can depend on time *t,* on the position of the particle *x,* and on the velocity υ_x. Consequently, in the general case Eq. (10.6) is, mathematically, a differential equation of the second order of the form

$$\frac{d^2x}{dt^2} = \Phi\left(t, x, \frac{dx}{dt}\right). \tag{10.9}$$

The equation can be solved for every specific problem after determining the form of its right-hand member, which depends on the applied forces. When Eq. (10.9) is integrated for a given problem, the general solution will include two constants of integration C_1 and C_2, and the *general solution* will be

$$x = f\left(t, C_1, C_2\right). \tag{10.10}$$

To solve a concrete problem, it is necessary to determine the values of the constants C_1 and C_2. For this we introduce the so-called *initial conditions.*

Investigation of any motion begins with some specified instant called the *initial time t* = 0, usually the moment when the motion under the action of the given forces starts. The position occupied by a particle at the initial time is called its *initial displacement,* and its velocity at that time is its *initial velocity* (a particle can have an initial velocity either because at time t = 0 it was moving under no force or because up to time t = 0 it was subjected to the action of some other forces). To solve the principal problem of dynamics, we must know, besides the applied forces, the *initial conditions,* i.e., the position and velocity of the particle at the initial time.

In the case of rectilinear motion, the initial conditions are specified in the form

$$\text{at } t = 0,\ x = x_0 \text{ and } \upsilon_x = \upsilon_0. \tag{10.11}$$

From the initial conditions we can determine the constants C_1 and C_2 and find the *partial solution* of Eq. (10.9), which gives the equation of motion of the particle:

$$x = f\left(t, x_0, \upsilon_0\right). \tag{10.12}$$

The following simple example will explain the above.

Let there be a force **Q** of constant magnitude and direction acting on a particle. Then Eq. (10.7) acquires the form

$$m\frac{d\upsilon_x}{dt} = Q_x.$$

As Q_x = const, multiplying both members of the equation by dt and integrating, we obtain:

$$\upsilon_x = \frac{Q_x}{m}t + C_1. \tag{10.13}$$

Substituting the value of υ_x into Eq. (10.7), we have:

$$\frac{dx}{dt} = \frac{Q_x}{m}t + C_1.$$

Multiplying by dt and integrating once again, we obtain:

$$x = \frac{1}{2}\frac{Q_x}{m}t^2 + C_1 t + C_2. \tag{10.14}$$

This is the general solution of Eq. (10.9) for the specific problem in the form given by Eq. (10.10).

Now let us determine the integration constants C_1 and C_2, assuming for the specific problem that the initial conditions are given by (10.11). Solutions (10.13) and (10.14) must satisfy any moment of time, including t = 0. Therefore, substituting zero for t in Eqs. (10.13) and (10.14), we should obtain υ_0 and x_0 instead of υ_x and x, i.e., we should have

$$C_1 = \upsilon_0, \quad C_2 = x_0.$$

These equations give the values of the constants C_1 and C_2 which satisfy the initial conditions of a given problem. Substituting these values into Eq. (10.14), we finally obtain the relevant equation of motion in the form expressed by Eq. (10.12):

$$x = x_0 + \upsilon_0 t + \frac{1}{2}\frac{Q_x}{m}t^2. \tag{10.15}$$

5. Solution of problems

Solution of problems of dynamics by integrating the differential equations of motion includes the following operations:

(1) Writing the differential equation of motion. For this,

(a) Choose an origin (usually coinciding with the initial position of the particle) and draw a coordinate axis along the line of motion, as a rule in the direction of motion; if, for the applied forces, a particle has a position of equilibrium, it is convenient to choose the origin to coincide with the position of static equilibrium.

(b) Depict the moving particle in an arbitrary position (but such that $x > 0$ and $\upsilon_x > 0$; the latter condition is important when the applied forces include forces depending on velocity), and draw all the forces acting on the particle.

(c) Compound the projections of all the forces on the coordinate axis and substitute the sum into the right side of the differential equation of motion. *It is important to express all the variable forces in terms of the quantities (t, x* or *v) on which they depend.*

(2) Integrating the differential equation of motion. The integration is carried out according to the rules of higher mathematics, depending on the form of the obtained equation, i.e., on the form of the right-hand member of Eq. (10.9). When besides the constant forces there is one variable force that depends *only* on time *t,* or *only* on distance *x,* or *only* on velocity υ, the equation of rectilinear motion can generally be integrated by the method of separating the variables. If only the velocity has to be determined, it is often possible to solve the problem by integrating either Eq. (10.7) or Eq. (10.8).

(3) Determining the constants of integration. In order to determine the constants of integration, it is necessary from the conditions of the problem to define the initial conditions in the form (10.11). The values of the constants are found from the initial conditions, and they can be determined directly after each integration.

If the differential equation of motion is an equation with separable variables, instead of introducing integration constants we can immediately evaluate the definite integrals on both sides of the equation over the appropriate range.

(4) Determining the required quantities and analysing the obtained results. In order to be able to analyse the solution and also to verify the dimensions, the whole solution should be carried out in the most general form (in letter notation), inserting the numerical data only in the final results.

These general rules also hold for curvilinear motion.

6. Curvilinear motion of a particle

Consider a free particle moving under the action of forces $\mathbf{F}_1$, ..., $\mathbf{F}_n$. Let us draw a fixed set of axes *Oxyz* (Fig. 10.2). Projecting both members of the equation (10.3) on these axes we obtain the *differential equations of curvilinear motion of a body* in terms of the projections on rectangular cartesian axes:

$$m\frac{d^2x}{dt^2}=\sum_k F_{kx},\quad m\frac{d^2y}{dt^2}=\sum_k F_{ky},\quad m\frac{d^2z}{dt^2}=\sum_k F_{kz}\,. \qquad (10.16)$$

As the forces acting on the particle may depend on time, the displacement or the velocity of the particle, then by analogy with Eq. (10.9), the right-hand members of Eq. (10.16) may contain the time t, the coordinates x, y, z of the particle, and the projections of its velocity υ_x, υ_y, υ_z. Furthermore, the right side of each equation may include all these variables.

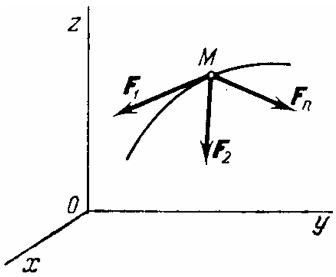


Fig. 10.2

Eqs. (10.16) can be used to solve both the first and the second (principal) problems of dynamics. To solve the principal problem of dynamics we must know, besides the acting forces, the initial conditions, i.e., the position and velocity of the particle at the initial time. The initial conditions for a set of coordinate axes $Oxyz$ are specified in the form

$$\text{at } t=0,\quad \left.\begin{aligned} x=x_0,\ y=y_0,\ z=z_0 \\ \upsilon_x=\upsilon_{x0},\ \upsilon_y=\upsilon_{y0},\ \upsilon_z=\upsilon_{z0}. \end{aligned}\right\}. \qquad (10.17)$$

Knowing the acting forces, by integrating Eq. (10.16) we find the coordinates *x, y, z* of the moving particle as functions of time *t*, i.e., the equation of motion for the particle. The solutions will contain six constants of integration C_1, C_2, …, C_6, the values of which must be found from the initial conditions (10.17).

LECTURE 11
RECTILINEAR VIBRATION OF A PARTICLE

The study of vibrations is essential for a number of physical and engineering fields. Although the vibrations studied in such different fields as mechanics, radio engineering, and acoustics are of different physical nature, the fundamental laws hold for all of them. The study of mechanical vibrations is therefore of importance not only because they are frequently encountered in engineering but also because the results obtained in investigating mechanical vibrations can be used in studying and understanding vibration phenomena in other fields.

1. Free vibrations neglecting resisting forces

We shall start with examining free vibration of a particle, neglecting resisting forces. Consider a particle M (Fig. 11.1) moving rectilinearly under the action of a single *restoring force* F directed towards a fixed centre O and proportional to the distance from that centre. The projection of F on axis Ox is

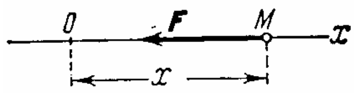


Fig. 11.1

$$F_x = -cx. \tag{11.1}$$

We see that the forced **F** tends to return the particle to its position of equilibrium O, where F=0, which is why it is called a "restoring" force. Examples of such a force are an elastic force and the force of attraction.

Let us derive the equation of motion of particle M. Writing the differential equation of motion, we obtain:

$$m\frac{d^2x}{dt^2} = -cx.$$

Dividing both sides of the equation by m and introducing the notation

$$\frac{c}{m} = k^2, \tag{11.2}$$

we reduce the equation to the form

$$\frac{d^2x}{dt^2} + k^2x = 0. \tag{11.3}$$

Eq. (11.3) is the *differential equation of free vibrations without resistance.* The solution of this linear homogeneous differential equation of the second order is sought in the form $x=e^{nt}$. Assuming $x=e^{nt}$ in Eq. (11.3), we obtain for the determination of n the so-called characteristic equation, which in the present case has the form $n^2–k^2=0$. As the solutions of this equation are purely imaginary ($n_{1,2}=\pm ik$), from the theory of differential equations the general solution of Eq. (11.3) has the form

$$x = C_1 \sin kt + C_2 \cos kt, \tag{11.4}$$

where C_1 and C_2 are constants of integration.

If we replace C_1 and C_2 by constants a and α, such that $C_1 = a\cos\alpha$ and $C_2 = a\sin\alpha$, we obtain

$$x = a\sin(kt + \alpha), \tag{11.5}$$

This is another form of the solution of Eq. (11.3) in which the constants of integration appear as a and a and which is more convenient for general analysis.

The velocity of a particle in this type of motion is

$$\upsilon_x = \frac{dx}{dt} = ak\cos(kt + \alpha). \tag{11.6}$$

The vibration of a particle described by Eq. (11.5) is called *simple harmonic motion.*

All the characteristics of this type of motion lend themselves to visual kinematic interpretation. Consider a particle B moving uniformly along a circle of radius a from a point B_0 defined by the angle $DOB_0 = \alpha$ (Fig. 11.2), and let the constant angular velocity of radius OB be k. Then, at any instant t angle $\varphi = \angle DOB = \alpha + kt$ and, it will be readily noticed, the projection M of point B on the diameter perpendicular to DE moves according to the law $x = a\sin(kt + \alpha)$, where $x = OM$, i.e., the projection performs harmonic motion.

The quantity a, which is the maximum distance of M from the centre of vibration, is called the *amplitude of vibration.* The quantity $\varphi = \alpha + kt$ is called the *phase of vibration.* Unlike the coordinate x, the phase φ defines both the position of the particle at any given time and the direction of its subsequent motion. For example, from position M at phase φ the particle will move to the right, at phase $(\pi - \varphi)$ it will move to the left. Phases differing by 2π are considered identical. The quantity α defines the *initial phase,* with which the motion begins. For example, at $\alpha = 0$ the motion is according to the sine law (it begins at O and the velocity is directed to the right); and at $\alpha = 0.5\pi$ y the motion is according to the cosine law (starting from point $x = a$ with a velocity $\upsilon_0 = 0$). The quantity k which coincides with the angular velocity of the rotating radius OB in Fig. 11.2 is called the *angular,* or *circular, frequency of vibration.*

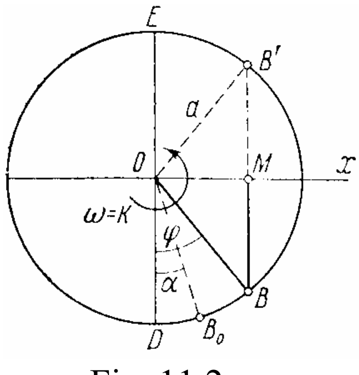


Fig. 11.2

The time T (or τ) in which the moving particle makes one complete vibraton is called the *period of vibration.* In one period the phase changes by 2π. Consequently, we must have $kT - 2\pi$, whence the period

$$T = \frac{2\pi}{k}. \tag{11.7}$$

The quantity ν, which is the inverse of the period and specifies the number of oscillations per second is called the frequency of vibration:

$$\nu = \frac{1}{T} = \frac{k}{2\pi}. \tag{11.8}$$

It can be seen from this that the quantity k differs from ν only by a constant multiplier 2π. Usually we shall speak of the quantity k as of frequency.

The values of a and α are determined from the initial conditions. Assuming that, at $t = 0$, $x = x_0$ and $\upsilon_x = \upsilon_0$, we obtain from Eqs. (11.5) and (11.6) $x_0 = a \sin \alpha$ and $\upsilon_0/k = a \cos \alpha$. By first squaring and adding these equations and then dividing them, we obtain

$$a = \sqrt{x_0^2 + \frac{\upsilon_0^2}{k^2}}, \quad \text{tg}\alpha = \frac{kx_0}{\upsilon_0}. \tag{11.9}$$

Note the following properties of free vibration without resistance:

(1) the amplitude and initial phase depend on the initial conditions;

(2) the frequency $k,$ and consequently the period $T,$ *do not depend* on the initial conditions [see Eqs. (11.2) and (11.7)] and are invariable characteristics for a given vibrating system.

It follows, in particular, that if a problem requires that only the period (or frequency) of vibration be determined, it is necessary to write a differential equation of motion in the form (11.3). Then T is found immediately from Eq. (11.7) without integrating.

2. Effect of a constant force on the free vibration of a particle

Let the particle M in Fig. 11.3 be subject, in addition to the restoring force F directed towards the centre $O,$ to a force P constant in magnitude and direction. The value of force F continues to be proportional to the distance from the centre $O,$ i.e., $F = -c\ OM.$ Obviously, in that case the equilibrium point is

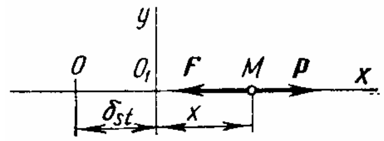


Fig. 11.3

Ox at a distance $OO_1 = \delta_{st}$ from O, given by the equation $c\delta_{st} = P$, or

$$\delta_{st} = \frac{P}{c}. \tag{11.10}$$

We shall call δ_{st} the *static deflection* of the particle.

Placing the origin of the reference system at O_1, direct axis O_1x in the direction of force **P**. Then $F_x = -c(x+\delta_{st})$, and $P_x = P$. Writing the differential equation of motion and taking into account that, by Eq. (11.10), $c\delta_{st} = P$, we have:

$$\frac{d^2x}{dt^2} + k^2x = 0.$$

The obtained equation, in which k is given by Eq. (11.2), is the same as Eq. (11.3). Hence we conclude that a constant force **P** does not affect the character of the vibrations of a particle under the action of a restoring force **F** and only displaces the centre of vibration in the direction of **P** by the amount of the static deflection δ_{st}.

Let us express the period of vibration in terms of δ_{st}. From (11.2) and (11.10), we have $k^2 = P/m\delta_{st}$. Then Eq. (11.7) gives:

$$T = 2\pi\sqrt{\frac{m}{P}\delta_{st}}\ . \tag{11.11}$$

Thus, *the period of vibration is in proportion to the square root of the static deflection* δ_{st}.

In particular, if P is the force of gravity, as in the case of vibration of a load on a vertical spring, then $P = mg$, and Eq. (11.11) takes the form

$$T = 2\pi\sqrt{\frac{\delta_{st}}{g}}\ . \qquad (11.11)$$

3. Free vibration with a resisting force proportional to velocity (damped vibration)

Let us see how the resistance of a surrounding medium affects vibrations, assuming the resisting force proportional to the first power of the velocity: $\mathbf{R} = -\mu\mathbf{v}$ (the minus indicates that force **R** is opposite to $\mathbf{v}$). Let a moving particle be acted upon by a restoring force **F** and a resisting force **R** (Fig. 11.4). Then $F_x = -cx$ and

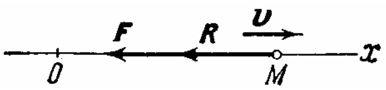


Fig. 11.4

$R_x = -\mu \upsilon_x = -\mu dx/dt$ and the differential equation of motion is

$$m\frac{d^2x}{dt^2} = -cx - \mu\frac{dx}{dt}.$$

Dividing both sides by m, we obtain:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0, \qquad (11.12)$$

where

$$\frac{c}{m} = k^2, \quad \frac{\mu}{m} = 2b. \qquad (11.13)$$

It is easy to verify that k and b have the same dimension (s^{-1}), which makes it possible to compare them.

Eq. (11.12) is the *differential equation of free vibrations with a resisting force proportional to the velocity.* Its solution, as in the case of Eq. (11.3), is sought in the form $x = e^{nt}$. Substituting into Eq. (11.13), we obtain the characteristic equation $n^2+2bn+k^2 = 0$, the roots of which are:

$$n_{1,2} = -b \pm \sqrt{b^2 - k^2}. \qquad (11.14)$$

Let us consider the case when $k > b$, i.e., when the resistance is small as compared with the restoring force. Introducing the notation

$$k_1 = \sqrt{b^2 - k^2}. \qquad (11.15)$$

from (11.14) we obtain $n_{1,2} = -b \pm ik_1$ i.e., the solutions of the characteristic equation are complex. In that case the general solution of Eq. (11.12) differs from the solution of Eq. (11.2) only by the multiplier e^{-bt}, i.e., it has the form

$$x = e^{-bt}\left(C_1 \sin k_1 t + C_2 \cos k_1 t\right), \qquad (11.16)$$

or, by analogy with Eq. (11.5),

$$x = ae^{-bt}\sin\left(k_1 t + \alpha\right). \qquad (11.17)$$

The quantities a and α are constants of integration and are determined by the initial conditions.

Vibrations according to the law (11.17) are called *damped* because, due to the multiplier e^{-bt}, the value of $x = OM$ decreases with time and tends to zero. A graph of such

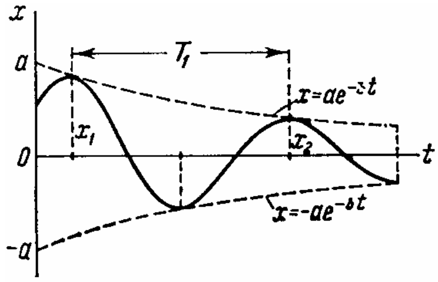


Fig. 11.5

vibrations is given in Fig. 11.5 [the curve lies between the broken curves $x = ae^{-bt}$ and $x = -ae^{-bt}$, as $\sin(k_1t+\alpha)$ cannot exceed unity].

The time T_1, equal to the period of $\sin(k_1t+\alpha)$, i.e., the quantity

$$T_1 = \frac{2\pi}{k_1} = \frac{2\pi}{\sqrt{b^2 - k^2}}. \qquad (11.18)$$

is conventionally called the *period of damped vibration.* In the course of one period the particle performs a complete vibration, e.g., having begun moving from position $x = 0$ to the right (see Fig. 11.4) it arrives at the same position, again moving to tbe right. Taking Eq. (11.7) into account, Eq. (11.18) can be written in the form

From the equations we see that $T_1 > T$, i.e., that resistance to vibration tends to increase the period of vibration. When, however, the resistance is small ($b \ll k$), the quantity b^2/k^2 can be neglected in comparison with unity, and we can assume $T_1 \approx T$. Thus, a small resistance has no practical effect on the period of vibration.

The time interval between two successive displacements of an oscillating particle to the right or to the left is also equal to T_1. Hence, if the first (maximum) displacement x_1 to the right takes place at time t_1, the second displacement x_2 will be at time $t_2 = t_1 + T_1$ etc. Then, by Eq. (11.17) and taking into account that $k_1T_1 = 2\pi$, we have:

$$x_1 = ae^{-bt_1}\sin\left(k_1t_1 + \alpha\right),$$
$$x_2 = ae^{-b(t_1+T_1)}\sin\left(k_1t_1 + k_1T_1 + \alpha\right) = x_1e^{-bT_1}.$$

Similarly, for any displacement x_{n+1} we shall have $x_{n+1}=x_ne^{-bT_1}$. Thus we find that the amplitude of vibration decreases in geometric progression. The ratio of this progression e^{-bT_1} is called the *decrement,* and the modulus of its logarithm, i.e., the quantity bT_1, the *logarithmic decrement.*

It follows from these results that a small resistance has practically no effect on the period of vibration but gradually damps it by virtue of the amplitude of vibration decreasing according to a law of geometric progression.

Let us consider the case when $b > k$, i.e., the resistance is large as compared with the restoring force. Introducing the notation $b^2-k^2=r^2$, we find that in this case the solutions of the characteristic equation (11.14) are $n_{1,2} = -b\pm r$, i.e., both are real and negative (as $r < b$). Consequently, when $b>k$ the solution of Eq. (11.12) describing the law of motion of the particle has the form

$$x = C_1 e^{-(b+r)t} + C_2 e^{-(b-r)t}.$$

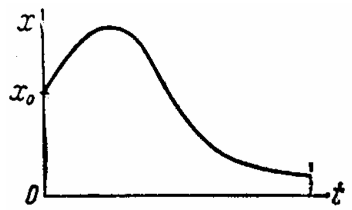


Fig. 11.6

Since with time the function e^{-at}, where $a > 0$, decreases gradually, tending to zero, the particle no longer vibrates but instead, under the influence of the restoring force, gradually approaches the position of equilibrium $x = 0$. A graph of such motion (if at $t = 0$, $x = x_0$ and $\upsilon_0 > 0$) has the form shown in Fig. 11.6.

4. Forced vibration. Resonance

Let us consider an important case of vibration where, in addition to a restoring force **F**, a particle is also subjected to a force **Q**, varying periodically with time, whose projection on axis *Ox* is

$$Q_x = Q_0 \sin pt. \tag{11.19}$$

This force is called a *disturbing force,* and the vibration caused by it is called *forced.* The quantity p in Eq. (11.19) is called the *frequency of the disturbing force.*

A disturbing force may vary with time according to other laws, but we shall consider only the case of Q_x defined by Eq. (11.19). This type of disturbing force is called a *periodic force.*

(1) Undamped Forced Vibration. Consider the motion of a particle on which, besides the restoring force **F**, is acting only a disturbing force **Q**. The differential equation of motion will be

$$m\frac{d^2x}{dt^2} = -cx + Q_0 \sin pt.$$

Divide both sides of the equation by m and assume

$$\frac{Q_0}{m} = P_0. \tag{11.20}$$

Then, taking into account the expression (11.2), the equation takes the form

$$\frac{d^2x}{dt^2} + k^2 x = P_0 \sin pt. \tag{11.21}$$

Eq. (11.21) is the *differential equation of undamped forced vibration* of a particle. From the theory of differential equations, its solution is

$x = x_1 + x_2$ where x_1 is the general solution of the equation without the right side, i.e., the solution of Eq. (11.3) as given by Eq. (11.5), and x_2 is a particular solution of the complete equation (11.21).

Assuming $p \neq k$, let us find the solution of x_2 in the form

$$x_2 = A \sin pt ,$$

where A is a constant such that Eq. (11.21) becomes an identity. Substituting the expression of x_2 and its second derivative into Eq. (11.21), we have:

$$-p^2 A \sin pt + k^2 A \sin pt = P_0 \sin pt .$$

This equation is satisfied at any t, if $A\ (k^2 - p^2) = P_0$, or

$$A = \frac{P_0}{k^2 - p^2} .$$

Thus, the required particular solution is

$$x_2 = \frac{P_0}{k^2 - p^2} \sin pt , \tag{11.22}$$

As $x = x_1 + x_2$ and the expression for xx is given by Eq. (11.5), the general solution of Eq. (11.21) takes the final form

$$x = a \sin\left(kt + \alpha\right) + \frac{P_0}{k^2 - p^2} \sin pt , \tag{11.23}$$

where a and α are constants of integration determined by the initial conditions.

Solution (11.23) shows that in the present case the vibration of a particle consists of (1) *free vibrations* of amplitude a (depending on the initial conditions) and frequency k called *natural vibrations* and (2) *forced vibrations* of amplitude A (not depending on the initial conditions) and frequency p.

In practice, due to the inevitable presence of various damping forces, the natural vibrations rapidly disappear. Therefore in this type of motion the forced vibrations defined by Eq. (11.22) are of primary importance.

Resonance. When $p = k$, i.e., when the frequency of the disturbing force equals the frequency of the natural vibrations, the phenomenon known as *resonance* occurs. The case is not covered by Eqs. (11.22), but it can be proved that when resonance takes place, the amplitude of forced vibration increases indefinitely, as shown below in Fig. 11.7.

At $p = k$, Eq. (11.21) does not contain the particular solution $x_2 = A\sin pt$, and the solution must be sought in the form

$$x_2 = Bt\cos pt.$$

From this we obtain the law of undamped forced vibrations when resonance occurs:

$$x = -\frac{P_0}{2p}t\sin(pt - \frac{\pi}{2}), \qquad (11.24)$$

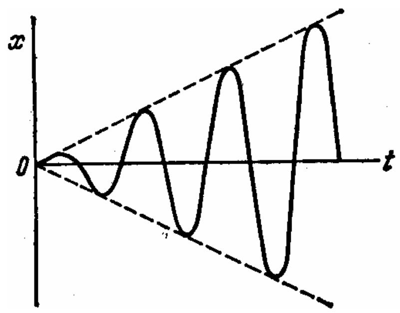


Fig. 11.7

We see that the amplitude of forced vibration during resonance does increase in proportion to time, and the law of vibration has the form shown in Fig. 11.7. The phase shift in resonance is $\pi/2$.

(2) Damped Forced Vibration. Consider the motion of a particle on which are acting a restoring force **F**, a damping force **R** proportional to the velocity, and a disturbing force **Q** given by Eq. (11.19). The differential equation of this motion has the form

$$m\frac{d^2x}{dt^2} = -cx - \mu\frac{dx}{dt} + Q_0\sin pt.$$

Dividing the equation by m and taking into account the expressions (11.13) and (11.20), we obtain:

$$m\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = Q_0\sin pt. \qquad (11.25)$$

Eq. (11.25) is the *differential equation of damped forced vibration* of a particle. Its general solution, as is known, has the form $x = x_1 + x_2$, where x_1 is the general solution of the equation without the right side, i.e., of Eq. (11.12) [at $k > b$ this solution is given by Eq. (11.17)], and x_2 is a particular solution of the complete equation (11.25). Let us find the solution x_2 in the form

$$x_2 = A\sin(pt - \beta),$$

where A and β are constants so chosen that Eq. (11.25) becomes an identity.

Substituting these expressions into the left side of Eq. (11.25) and introducing for the sake of brevity the notation $pt - \beta = \psi$, we obtain:

$$A(-p^2 + k^2)\sin\psi + 2bpA\cos\psi = P_0(\cos\beta\sin\psi + \sin\beta\cos\psi).$$

For this equation to be satisfied at any value of ψ, i.e., at any instant of time, the factors of $\sin\psi$ and $\cos\psi$ in the left and right sides should be separately equal. Hence,

$$A(k^2 - p^2) = P_0 \cos\beta, \quad 2bpA = P_0 \sin\beta.$$

First squaring and adding these equations (they are also used to determine β uniquely) and then dividing one by the other, we obtain:

$$A = \frac{P_0}{\sqrt{(k^2 - p^2)^2 + 4b^2p^2}}, \quad \mathrm{tg}\beta = \frac{2bp}{k^2 - p^2}. \tag{11.26}$$

As $x = x_1 + x_2$, and the expression for x_1 (when $k>b$) is given by Eq. (11.17), we have the final solution of Eq. (11.25) in the form

$$x = Ae^{-bt}\sin\left(k_1 t + \alpha\right) + B\sin\left(pt - \beta\right) \tag{11.27}$$

where a and α are constants of integration determined by the initial conditions, and the expressions for A and β are given by Eqs. (11.26) and do not depend on the initial conditions. For b=0 the solution (11.27) is just (11.22) and (11.23) for the case without resisting forces.

These vibrations are compounded of *natural* vibration [the first term in Eq. (11.27); Fig. 11.8*a*] and *forced* vibration [the second term in Eq. (11.27); Fig. 11.8*b*]. It was established that it is transient and is damped fairly quickly, and after a certain interval of time t_{tr}, called the *transient period,* can be neglected.

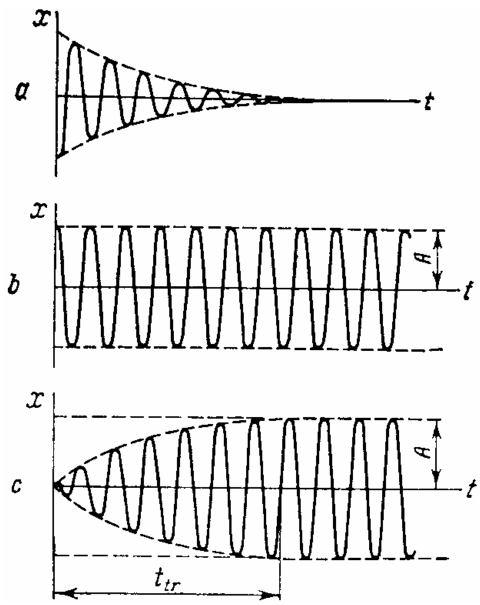


Fig. 11.8

If, for example, we assume that free vibrations can be neglected from the moment when their amplitude is less than 0.01 A, then the value of t_{tr} can be determined from the equation $ae^{-bt} = 0.01A$,

$$t_{tr} = \frac{1}{b}\ln\frac{100a}{A}. \tag{11.28}$$

We see, thus, that the less the resistance (i.e., the less the value of b), the greater the transient period.

A possible picture of transient vibration according to the law (11.27) and starting from rest, is shown in Fig. 11.8*c*. Given other initial conditions and ratios of the frequencies p and k_1, the character of the vibrations in the time interval $0 < t < t_{tr}$ can be quite different.

However, in all cases, after the transient period elapses the natural vibrations will, for all practical purposes, cease and the particle will vibrate according to the law

$$x_2 = A\sin(pt - \beta). \tag{11.29}$$

This is *steady-state forced vibration,* a sustained periodic motion with an amplitude A denned by Eq. (11.26) and a frequency p equal to the frequency of the disturbing force. The quantity β characterises the phase shift of forced vibration with respect to the disturbing force.

LECTURE 12
INTRODUCTION TO THE DYNAMICS OF A SYSTEM. MOMENTS OF INERTIA OF RIGID BODIES
1. Mechanical systems. External and internal forces

A mechanical system is defined as such a collection of material points (particles) or bodies in which the position or motion of each particle or body of the system depends on the position and motion of all the other particles or bodies. We shall regard a body as a system of its particles.

External forces are defined as the forces exerted on the members of a system by particles or bodies not belonging to the given system. *Internal forces* are defined as the forces of interaction between the members of the same system. We shall denote external forces by the symbol $\mathbf{F}^{(e)}$, and internal forces by the symbol $\mathbf{F}^{(i)}$.

Both external and internal forces can be either active forces or reactions of constraints. The division of forces into external and internal is purely relative, and it depends on the extent of the system whose motion is being investigated. In considering the motion of the solar system as a whole, for example, the gravitational attraction of the sun acting on the earth is an internal force; in investigating the earth's motion about the sun, the same force is external.

Internal forces possess the following properties:

(1) The geometrical sum (the principal vector) of all the internal forces of a system is zero. This follows from the third law of dynamics, which states that any two particles of a system (Fig. 12.1) act on each other with

equal and oppositely directed forces $\mathbf{F}^{(i)}{}_{12}$ and $\mathbf{F}^{(i)}{}_{21}$, the sum of which is zero. Since the same is true for any pair of particles of a system,

$$\sum_k \mathbf{F}_k^{(i)} = 0 .$$

(2) The sum of the moments (the principal moment) of all the internal forces of a system with respect to any centre or axis is zero. For if we take an arbitrary centre O, it is apparent from Fig. 12.1 that $\mathbf{m}_O(\mathbf{F}^{(i)}{}_{12})+\mathbf{m}_O(\mathbf{F}^{(i)}{}_{21})=0$. The same result holds for the moments about any axis. Hence, for the system as a whole we have:

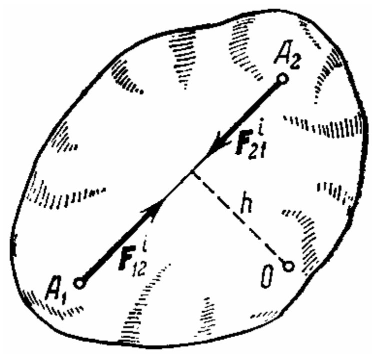


Fig. 12.1

$$\sum_k \mathbf{m}_O(\mathbf{F}_k^{(i)}) = 0 \quad \text{or} \quad \sum_k m_x(\mathbf{F}_k^{(i)}) = 0 .$$

It does not follow from the above, however, that the internal forces are mutually balanced and do not affect the motion of the system, for they are applied to *different* particles or bodies and may cause their mutual displacement. The internal forces will be balanced only when a given system is a rigid body.

2. Mass of a system. Centre of mass

The motion of a system depends, besides the acting forces, on its total mass and the distribution of this mass. The *mass of a system* is equal to the arithmetical sum of the masses of all the particles or bodies comprising it:

$$M = \sum_k m_k .$$

In a homogeneous field of gravity, where $g = \text{const.}$, the weight of every particle of a body is proportional to its mass, hence the distribution of mass can be judged according to the position of the centre of gravity. Let us rewrite the equations defining the coordinates of the centre of gravity in a form manifestly including mass. Cancelling out g, we obtain:

$$x_C = \frac{1}{M}\sum_k m_k x_k, \quad y_C = \frac{1}{M}\sum_k m_k y_k, \quad z_C = \frac{1}{M}\sum_k m_k z_k . \qquad (12.1)$$

The equations include only the masses m_k of the material points (particles) of the body and their coordinates x_k, y_k, z_k. Hence, the position of point C (x_C, y_C, z_C) gives the distribution of mass in the body or in any mechanical system, where m_k and x_k, y_k, z_k are the masses and coordinates of the system's respective points.

The geometric point C whose coordinates are given by Eqs. (12.1) is called the *centre of mass,* or *centre of inertia* of a mechanical system.

If the position of the centre of mass is defined by its radius vector $\boldsymbol{r}_C$, we can obtain from Eqs. (12.1) the following expression:

$$\mathbf{r}_C = \frac{1}{M}\sum_k m_k \mathbf{r}_k \,. \tag{12.2}$$

where $\boldsymbol{r}_k$ *is* the radius vector of particle k of the system.

3. Moment of inertia of a body about an axis. Radius of gyration

The position of centre of mass does not characterise completely the distribution of mass in a system. For if in the system in Fig. 12.2 the distance h of each of two identical spheres A and B from the axis Oz is increased by the same quantity, the location of the centre of mass will not change, though the distribution of mass will change and influence the motion of the system (all other conditions remaining the same, the rotation about axis Oz will be slower).

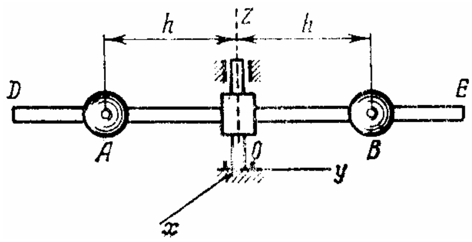


Fig. 12.1

Accordingly, another characteristic of the distribution of mass, called the moment of inertia, is introduced in mechanics. *The moment of inertia of a body* (*system*) *with respect to a given axis Oz (or the axial moment of inertia) is defined as the quantity equal to the sum of the masses of the particles of the body (system) each multiplied by the square of its perpendicular distance from the axis*:

$$J_z = \sum_k m_k h_k^2 \,. \tag{12.3}$$

It follows from the definition that the moment of inertia of a body (or system) with respect to any axis is always positive.

It will be shown further on that axial moment of inertia plays the same part in the rotational motion of a body as mass does in translational motion, i.e., *moment of inertia is a measure of a body's inertia in rotational motion.*

By Eq. (12.3), the moment of inertia of a body is equal to the sum of the moments of inertia of all its parts with respect to the same axis.

For a material point located at a distance h from an axis, $J_z = mh^2$. The unit for the moment of inertia in the SI system is 1 kg m^2.

In computing the axial moments of inertia the distances of the points from the axes can be expressed in terms of their coordinates x_k, y_k, z_k. Then the moments of inertia about the axes $Oxyz$ will be given by the following equations:

$$J_x = \sum_k m_k (y_k^2 + z_k^2),\ J_y = \sum_k m_k (x_k^2 + z_k^2),\ J_z = \sum_k m_k (y_k^2 + x_k^2). \quad (12.4)$$

The concept of the *radius of gyration* is often employed in calculations. The radius of gyration of a body with respect to an axis Oz is a linear quantity i_z defined by the equation

$$J_z = J_z = Mi_z^2. \quad (12.5)$$

where M is the mass of the body.

It follows from the definition that geometrically the radius of gyration is equal to the distance from the axis Oz to a point, such that if the mass of the whole body were concentrated in it, the moment of inertia of the point would be equal to the moment of inertia of the whole body.

Knowing the radius of gyration, we can obtain the moment of inertia of a body from Eq. (12.5) and vice versa.

Eqs. (121.3) and (12.4) are valid for both rigid bodies and systems of material points. In the case of a solid body, dividing it into elementary parts, we find that in the limit the sum in Eq. (12.3) becomes an integral. Hence, taking into account that $dm = \rho dV$, where ρ is the density and V the volume, we obtain:

$$J_z = \int_{(V)} h^2 dm = \int_{(V)} \rho h^2 dV. \quad (12.6)$$

Eq. (12.6) are useful in calculating the moments of inertia of homogeneous bodies of geometric shape. As in that case the density ρ is constant, it can be taken out of the integral sign.

Let us determine the moments of inertia of some homogeneous bodies.

(1) Thin homogeneous rod of length *l* and mass *M*:

$$J_z = \frac{Ml^2}{3}. \quad (12.7)$$

(2) Thin circular homogeneous ring of radius *R* and mass *M*:

$$J_z = MR^2. \quad (12.8)$$

(3) Circular homogeneous disc or cylinder of radius *R* and mass *M*:

$$J_z = \frac{MR^2}{2}. \qquad (12.9)$$

(4) Uniform rectangular lamina of mass *M* with sides of length *a* and *b* (axis *x* in coincident with side *a*, axis *y* with side *b*):

$$J_x = \frac{Mb^2}{3}, \quad J_y = \frac{Ma^2}{3}. \qquad (12.10)$$

(5) Uniform right circular cone of mass *M* and base radius *R* (axis *z* is coincident with the axis of the cone):

$$J_z = 0.3MR^2. \qquad (12.11)$$

(6) Uniform sphere of mass *M* and base radius *R* (axis *z* is coincident with a diameter):

$$J_z = 0.4MR^2. \qquad (12.12)$$

4. Moments of inertia of a body about parallel axes. The parallel axis (Huygens') theorem

In the most general case, the moments of inertia of the same body with respect to different axes are different. Let us see how to determine the moment of inertia of a body with respect to any axis if its moment of inertia with respect to a parallel axis through the body is known.

Draw through the centre of mass of a body C arbitrary axes $Cx'y'z'$, and through an arbitrary point O on axis Cx' axes $Oxyz$, so that $Oy \parallel Cy'$ and $Oz \parallel Cz'$ (Fig. 12.3). Denoting the distance between axes Cz' and Oz by d, from Eqs. (12.4)

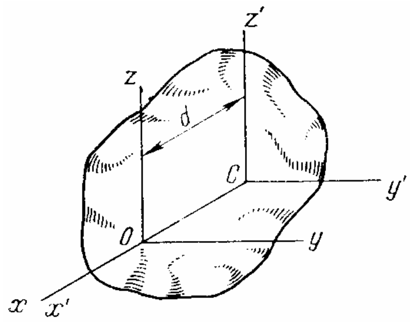


Fig. 12.3

$$J_{Oz} = \sum_k m_k (x_k^2 + y_k^2),$$

$$J_{Oz'} = \sum_k m_k (x_k'^2 + y_k'^2).$$

But it is apparent from the drawing that for any point of the body $x_k = x'_k - d$, and $y_k = y'_k$. Substituting these expressions for x_k and y_k into the expression for J_{Oz} and taking the common multipliers d^2 and $2d$ outside the parentheses, we obtain:

$$J_{Oz} = \sum_k m_k (x_k'^2 + y_k'^2) + d^2 \sum_k m_k - 2d \sum_k m_k x_k' \ .$$

The first summation in the right member of the equation is equal to $J_{Cz'}$, and the second to the mass M of the body. Let us find the value of the third summation. From Eq. (12.1) we know that, for the coordinates of the centre of mass

$$\sum_k m_k x_k' = M x_C' \ .$$

But since in our case point C is the origin, $x'_C = 0$, and consequently

$$\sum_k m_k x_k' = 0 .$$

We finally obtain

$$J_{Oz} = J_{Cz'} + Md^2 . \tag{12.13}$$

Eq. (12.13) expresses the **parallel axis theorem** enunciated by Huygens: *The moment of inertia of a body with respect to any axis is equal to the moment of inertia of the body with respect to a parallel axis through the centre of mass of the body plus the product of the mass of the body and the square of the distance between the two axes.*

It follows from Eq. (12.13) that $J_{Oz} > J_{Cz'}$. Consequently, of all the axes of same direction, the moment of inertia is least with respect to the one through the centre of mass.

LECTURE 13
THEOREM OF THE CHANGE IN THE KINETIC ENERGY OF A SYSTEM.
1. The kinetic energy of a particle

The kinetic energy of a particle is defined as a quantity equal to half the product of the mass of the particle and the square of its velocity

$$K = \frac{m\upsilon^2}{2} . \tag{13.1}$$

The units of measurement of this quantity are:

(a) In the SI system

kg m^2/s^2;

(b) In the mkg(f)s system

(kgf s^2/m) (m^2/s^2)= kgf m;

2. Work done by a force. Power

The concept of *work* is introduced as a measure of the action of a force on a body in a given displacement, specifically that action which is represented by the change in the magnitude of the velocity of a moving particle.

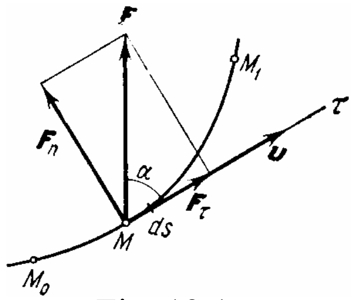


Fig. 13.1

First let us introduce the concept of elementary work done by a force in an infinitesimal displacement *ds*. The elementary work done by a force **F** (Fig. 13.1) is defined as a scalar quantity

$$dA = F_\tau ds . \qquad (13.2)$$

where F_τ is the projection of the force on the tangent to the path in the direction of the displacement, and *ds* is an infinitesimal displacement of the particle along that tangent.

This definition corresponds to the concept of work as a characteristic of that action of a force which tends to change the magnitude of velocity. For if force **F** is resolved into components $\mathbf{F}_\tau$ and $\mathbf{F}_n$, only the component $\mathbf{F}_\tau$, which imparts the particle its tangential acceleration, will change the magnitude of the velocity. As for component $\mathbf{F}_n$, it either changes the direction of the velocity vector υ (gives the particle its normal acceleration) or, in the case of constrained motion, changes the pressure on the constraint. Component $\mathbf{F}_n$ does not affect the magnitude of the velocity, or as they say, force $\mathbf{F}_n$ "does no work".

Noting that $F_\tau = F\cos\alpha$, we further obtain from Eq. (13.2):

$$dA = F\cos\alpha ds . \qquad (13.3)$$

Thus, *the elementary work done by a force is equal to the product of the projection of that force on the direction of displacement of the particle and the infinitesimal displacement ds* (Eq. 13.2) or, *the elementary work done by a force is the product of the magnitude of that force, the infinitesimal displacement ds, and the cosine of the angle between the direction of the force and the direction of the displacement* (Eq. 13.3).

If angle α is acute, the work is of positive sense. In particular, at $\alpha=0$, the elementary work $dA=Fds$.

If angle α is obtuse, the work is of negative sense. In particular, at $\alpha=180°$, the elementary work $dA=-Fds$.

If angle $\alpha=90°$, i.e., if a force is directed perpendicular to the displacement, the elementary work done by the force is zero.

The sign of the work has the following meaning: the work is positive when the tangential component of the force is pointed in the direction of the displacement, i.e., when the force accelerates the motion; the work is negative when the tangential component is pointed opposite the displacement, i.e., when the force retards the motion.

As we know from kinematics, the vector of the elementary displacement of a particle $d\mathbf{r}=\boldsymbol{\upsilon}dt$, and $ds=|\boldsymbol{\upsilon}|dt$, whence $ds=|d\mathbf{r}|$. Using the concept of the scalar product of two vectors employed in vector algebra, Eq. (13.3) can be represented in the form

$$dA = \mathbf{F}d\mathbf{r}\,. \tag{13.4}$$

Consequently, *the elementary work done by a force equals the scalar product of the force vector and the vector of the elementary displacement of its point of application*.

Lot us now find the analytical expression for elementary work. For this we resolve force **F** into components F_x, F_y, F_z parallel to the coordinate axes. The infinitesimal displacement ds is compounded of the displacements dx, dy, dz parallel to the coordinate axes, where x, y, z are the coordinates of point. The work done by force **F** in the displacement ds can be calculated as the sum of the work done by its components F_x, F_y, F_z in the displacements dx, dy, dz. But the work in the displacement dx is done only by component F_x and is equal to $F_x dx$. The work in the displacements dy and dz is calculated similarly. Thus, we finally obtain

$$dA = F_x dx + F_y dy + F_z dz\,. \tag{13.5}$$

Eq. (13.5) gives the *analytical expression of the elementary work* done by a force.

Eq. (13.5) can be obtained directly from (13.4) if the scalar product is expressed in terms of the projections of the vectors. Then, taking into account that the projections of the radius vector **r** of point M on the axes $Oxyz$ are equal to its cartesian coordinates x, y, z, we obtain at once $dA=F_x dx+F_y dy+F_z dz$.

The work done by a force in any finite displacement M_0M_1 (see Fig. 13.1) is calculated as the integral sum of the corresponding elementary works and is equal to

$$A_{(M_0M_1)} = \int_{M_0}^{M_1} F_\tau ds\,, \tag{13.6}$$

or

$$A_{(M_0M_1)} = \int_{M_0}^{M_1} (F_x dx + F_y dy + F_z dz)\,. \tag{13.7}$$

Thus, *the work done by a force in any displacement M_0M_1 to the integral of the elementary work taken along this displacement*.

The limits of the integral correspond to the values of the variables of integration at points M_0 and M_1 (or, more exactly, the integral is taken along the curve M_0M_1, i.e., it is curvilinear).

If the quantity F_τ is constant (**F_τ=const**), then from Eq. (13.6), denoting the displacement M_0M_1 by the symbol s_1, we obtain:

$$A_{(M_0M_1)} = F_\tau s_1\,. \tag{13.8}$$

In particular, such a case is possible when the acting force is constant in magnitude and direction (F=const.) and the point of application is in rectilinear motion (Fig. 252). In this case $F_\tau = F\cos\alpha$=const, and the work done by the force

$$A_{(M_0M_1)} = F s_1 \cos\alpha\,. \tag{13.9}$$

The unit of work in the SI system is the *joule* (1J = lN m), and in the mkg(f)s system, the kgf m.

Power. The term *power* is defined as the work done by a force in a unit of time (the time rate of doing work). If work is done at a constant rate, the power

$$W = \frac{A}{t_1}, \tag{13.10}$$

where t_1 ia the time in which the work A is done. In the general case,

$$W = \frac{dA}{dt} = \frac{F_\tau ds}{dt} = F_\tau \upsilon\,. \tag{13.11}$$

Thus, power is product of the tangential component of a force by the velocity.

The unit of power in the SI system is the *watt* (1 W=1 J/s), and in the mkg(f)s system, the kgf m/s. In engineering the unit of power commomly used is horsepower (hp), which is equal to 75 kgf m/s, or 736 W.

The work done by a machine can be expressed as the product of its power and the time of work. This has given rise to the commonly used technical unit of work, the kilowatt-hour (1 kW h=3.6 10^6 J≈367100 kgf m).

It can be seen from the equation $W=F_\tau \upsilon$ that if a motor has a given power W, the tractive force F_τ is inversely proportional to the velocity υ. That is why, for instance, on an upgrade or poor road an automobile goes into lower gear, thereby reducing the speed and developing a greater tractive force with the same power.

3. Examples of calculation of work

(1) Work done by gravity. Let a particle M subjected to the force of gravity **P** move from a point M_0 (x_0, y_0, z_0) to a point M_1 (x_1, y_1, z_1). Choose a coordinate system so that the axis Oz points vertically up (Fig. 13.2). Then P_x=0, P_y=0, P_z=-P. Substituting these expressions into Eq. (13.5) and taking into account that the integration variable is z, we obtain:

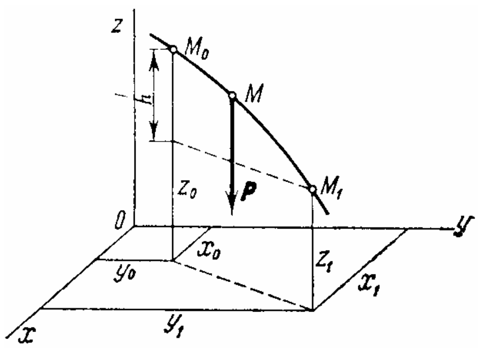


Fig. 13.2

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-P)dz = -P\int_{z_0}^{z_1} dz = P(z_0 - z_1). \qquad (13.12)$$

If point M_0 is higher than M_1 then z_0–z_1=h, where h is the vertical displacement of the particle; if M_0 is below M_1 then z_0–z_1=–(z_1– z_0)=–h. Finally we have:

$$A_{(M_0M_1)} = \pm Ph. \qquad (13.13)$$

Thus, *the work done by gravity is equal to the product of the magnitude of the force and the vertical displacement of the point to which it is applied, taken with the appropriate sign*. The work is positive if the initial point is higher than the final one and negative if it is lower.

It follows from this that the work done by gravity does not depend on the path along which the point of its application moves. Forces possessing this property are called *conservative forces*.

(2) Work Done by an Elastic Force. Consider a weight M lying in a horizontal plane and attached to the free end of a spring (Fig. 13.3).

Let point O on the plane represent the position of the end of the spring when it is not in tension ($AO=l_0$ is the length of the unextended spring) and let it be the origin of our coordinate system. Now if we draw the weight from its position of equilibrium O, stretching the spring to length l, acting on the weight will be the elastic force of the spring **F** directed towards O. According to Hooke's law, the magnitude of this force is proportional to the extension of the spring $\Delta l=l-l_0$. As in our case $\Delta l=x$, in magnitude

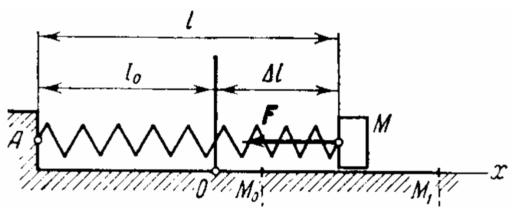


Fig. 13.3

$$F = c|\Delta l| = c|x|. \tag{13.14}$$

The factor c is called the *stiffness* of the spring, or the spring constant, and its dimension is $[c]$ = kgf/cm. Numerically, the stiffness c is equal to the force required to extend the spring by 1 cm.

Let us find the work done by the elastic force in the displacement of the weight from position $M_0(x_0)$ to position $M_1(x_1)$. As in this case $F_x=-F=-cx$, $F_y=F_z=0$, substituting these expressions into Eq. (13.7) we obtain:

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-cx)dx = -c\int_{x_0}^{x_1} xdx = \frac{c}{2}(x_0^2 - x_1^2). \tag{13.15}$$

In the obtained formula x_0 is the initial extension of the spring Δl_{in}, and x_1 is the final extension Δl_{fin}. Hence,

$$A_{(M_0M_1)} = \frac{c}{2}\left[(\Delta l_{in})^2 - (\Delta l_{fin})^2\right], \tag{13.16}$$

i.e., *the work done by an elastic force is equal to half the product of the stiffness and the difference between the squares of the initial and final extensions (or compressions) of a spring*.

The work is positive if $|\Delta l_{in}| > |\Delta l_{fin}|$, i.e., when the end of the spring moves towards the position of equilibrium, and negative when $|\Delta l_{in}| < |\Delta l_{fin}|$, i.e., when the end of the spring moves away from the position of equilibrium.

It can be proved that Eq. (13.16) holds for the case when the displacement of point M is not rectilinear. It follows, therefore, that the work done by the force **F** depends only on the quantities Δl_{in} and Δl_{fin} and does not depend on the actual path travelled by M. Consequently, an elastic force is also a *conservative force*.

(3) Work Done by Friction. Consider a particle moving on a rough surface or a rough curve. The magnitude of the frictional force acting on the particle is fN, where f is the coefficient of friction and N is the normal reaction of the surface. Frictional force is directed opposite to the displacement of the particle, whence $F_{fr\,x}$=–fN, and from Eq. (13.6),

$$A_{(M_0M_1)} = \frac{c}{2}\left[(\Delta l_{in})^2 - (\Delta l_{fin})^2\right], \tag{13.17}$$

If the friction force is constant, then A=–$F_{fr}s$, where s is the length of the arc M_0M_1 along which the particle moves.

Thus, *the work done by kinetic friction is always negative*. It depends on the length of the arc M_0M_1 and consequently *friction is a non-conservative force*.

4. Theorem of the change in the kinetic energy of a particle

Consider a particle of mass m displaced by acting forces from a position M_0 where its velocity is υ_0 to a position M_1 where its velocity is υ_1.

To obtain the required relation, consider the equation

$$m\boldsymbol{a} = \sum_k \mathbf{F}_k\,, \tag{13.18}$$

which expresses the fundamental law of dynamics. Projecting this equation on the tangent M_τ to the path of the particle in the direction of motion, we obtain:

$$ma_\tau = \sum_k F_{k\tau}\,. \tag{13.19}$$

The tangential acceleration in the left side of the equation can be written in the form

$$a_\tau = \frac{d\upsilon}{dt} = \frac{d\upsilon}{ds}\frac{ds}{dt} = \frac{d\upsilon}{ds}\upsilon\,, \tag{13.20}$$

whence, we have:

$$m\frac{d\upsilon}{ds}\upsilon=\sum_k F_{k\tau}\ . \qquad (13.21)$$

Multiplying both sides of the equation by ds, bring m under the differential sign. Then, notingthat $F_{k\tau}ds=dA_k$, where dA_k is the elementary work done by the force $\mathbf{F}_k$, we obtain an expression of the **theorem of the change in kinetic energy in differential form**:

$$d\left(\frac{m\upsilon^2}{2}\right)=\sum_k dA_k\ . \qquad (13.22)$$

Integrating both parts in the limits of corresponding values of the variables at points M_0 and M_1 we finally obtain:

$$\frac{m\upsilon_1^2}{2}-\frac{m\upsilon_0^2}{2}=\sum_k A_{(M_0M_1)}\ . \qquad (13.23)$$

Eq. (13.23) states the **theorem of the change in the kinetic energy of a particle in the final form**: *The change in the kinetic energy of a particle in any displacement is equal to the algebraic sum of the work done by all the forces acting on the particle in the same displacement.*

LECTURE 14
THEOREM OF THE CHANGE IN THE KINETIC ENERGY OF A SYSTEM (continuation).
5. Kinetic energy of a system

The kinetic energy of a system is defined as a scalar quantity T equal to the arithmetical sum of the kinetic energies of all the particles of the system:

$$K=\sum_k\frac{m_k\upsilon_k^2}{2}\ . \qquad (13.24)$$

Kinetic energy is a characteristic of both the translational and rotational motion of a system, which is why the theorem of the change in kinetic energy is so frequently used in problem solutions.

If a system consists of several bodies, its kinetic energy is, evidently, equal to the sum of the kinetic energies of all the bodies:

$$K=\sum_k K_k\ . \qquad (13.25)$$

Let us develop the equations for computing the kinetic energy of a body in different types of motion.

(1) Translational Motion. In this case all the points of a body have the same velocity, which is equal to the velocity of the centre of mass. Therefore, for any point k we have $\upsilon_k=\upsilon_C$, and Eq. (13.24) gives:

$$K_{trans}=\sum_k \frac{m_k\upsilon_C^2}{2}=\frac{1}{2}\left(\sum_k m_k\right)\upsilon_C^2,$$

or

$$K_{trans}=\sum_k \frac{m_k\upsilon_C^2}{2}=\frac{1}{2}M\upsilon_C^2, \tag{13.26}$$

Thus, in translational motion, the kinetic energy of a body is equal to half the product of the body's mass and the square of the velocity of the centre of mass. The value of K does not depend on the direction of motion.

(2) Rotational Motion. The velocity of any point of a body rotating about an axis Oz is $\upsilon_k=\omega h_k$, where h_k is the distance of the point from the axis of rotation and co is the angular velocity of the body. Substituting this expression into Eq. (13.24) and taking the common multipliers outside of the parentheses, we obtain:

$$K_{rotation}=\sum_k \frac{m_k\omega^2 h_k^2}{2}=\frac{1}{2}\left(\sum_k m_k h_k^2\right)\omega^2,$$

The term in the parentheses is the moment of inertia of the body with respect to axis z. Thus, we finally obtain:

$$K_{rotation}=\frac{1}{2}J_z\omega^2, \tag{13.27}$$

i.e., in rotational motion, the kinetic energy of a body is equal to half the product of the body's moment of inertia with respect to the axis of rotation and the square of its angular velocity. The value of K does not depend on the direction of the rotation.

(3) Plane Motion. In plane motion, the velocities of all the points of a body are at any instant directed as if the body were rotating about an axis perpendicular to the plane of motion and passing through the instantaneous centre of zero velocity C_V. Hence, by Eq. (13.27),

$$K_{plane}=\frac{1}{2}J_{C_V}\omega^2, \tag{13.28}$$

where $J_{C\mathrm{V}}$ is the moment of inertia of the body with respect to the instantaneous axis of rotation, and co is the angular velocity of the body.

The quantity $J_{C\mathrm{V}}$ in Eq. (13.27) is variable, as the position of the centre C_V continuously changes with the motion of the body. Let us introduce instead of $J_{C\mathrm{V}}$ a constant moment of inertia J_C with respect to an axis through the centre of mass C of the body. By the parallel axis theorem $J_{C\mathrm{V}}=J_C+Md^2$, where $d=PC$. Substituting this expression for $J_{C\mathrm{V}}$ into Eq. (13.28) and taking into account that point C_V is the instantaneous centre of zero velocity and therefore $\omega d=\omega PC=\upsilon_C$, where υ_C is the velocity of the centre of mass, we obtain finally:

$$K_{plane} = \frac{1}{2} M \upsilon_C^2 + \frac{1}{2} J_C \omega^2, \qquad (13.29)$$

Thus, in plane motion, the kinetic energy of a body is equal to the kinetic energy of translation of the centre of mass plus the kinetic energy of rotation relative to the centre of mass.

6. Some case of computation of work

(1) Work done by forces applied to a rotating body. The elemental work done by the force **F** applied to the body in Fig. 13.4 will be

$$dA = F_\tau ds = F_\tau h d\varphi,$$

since $ds=hd\varphi$, where $d\varphi$ is the angle of rotation of the body.

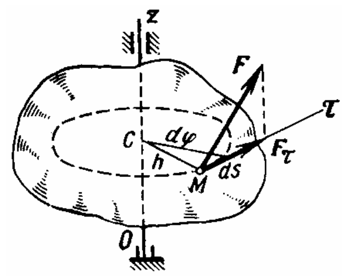


Fig. 13.4

But it is evident that $F_\tau h=m_z(\mathbf{F})$. We shall call the quantity $M_z= m_z(\mathbf{F})$ the *turning moment*, or *torque*. Thus we obtain:

$$dA = M_z d\varphi, \qquad (13.30)$$

i.e., the elemental work in this case is equal to the product of the torque and the elemental angle of rotation. Eq. (13.30) is valid when several forces are acting if it is assume that

$$M_z = \sum_k m_z(\mathbf{F}_k).$$

The work done in a turn through a finite angle φ_1 will be

$$A = \int_0^{\varphi_1} M_z d\varphi, \qquad (13.31)$$

and, for a constant torque (M_z=const.),

$$A = M_z \varphi_1 . \qquad (13.32)$$

If acting on a body is a force couple lying in a plane normal to Oz, then, evidently, M_z in Eqs. (13.30)-(13.32) will denote the moment of that couple.

Let us see how power is determined in this case. From Eq. (13.30) we find:

$$W = \frac{dA}{dt} = \frac{M_z d\varphi}{dt} = M_z \omega .$$

Thus, the power developed by forces acting on a rotating body is equal to the product of the torque and angular velocity of the body. For the same power, the torque increases as the angular velocity decreases.

(2) Work done by frictional forces acting on a rolling body. A wheel of radius R (Fig. 13.5) rolling without slipping on a plane (surface) is subjected to the action of a frictional force $\mathbf{F}_{\mathrm{fr}}$, which prevents the slipping of the point of contact B on the surface. The elemental work done by this force is $dA = -F_{\mathrm{tr}} ds_B$. But point B is the instantaneous centre of velocity, and $\upsilon_B = 0$. As $ds_B = \upsilon_B dt$, $ds_B = 0$, and for every elemental displacement $dA = 0$.

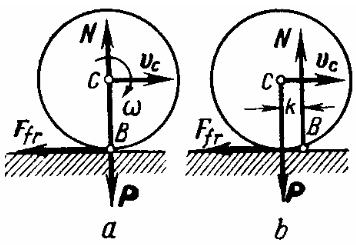


Fig. 13.5

Thus, in rolling without slipping, the work done by the frictional force preventing slipping is zero in any displacement of the body. For the same reason, the work done by the normal reaction **N** is also zero, assuming the body to be non-deformable and force **N** applied at point B, as shown in Fig. 13.5*a*.

The resistance to rolling due to deformation of the surfaces (Fig. 13.5*b*) creates a couple (**N**, **P**) with a moment $M = kN$, where k is the coefficient of rolling friction. Then by Eq. (13.30) and taking into account that the angle of rotation of a rolling wheel is

$$d\varphi = \frac{ds_C}{R},$$

we obtain:

$$dA_{roll} = -kNd\varphi = -\frac{k}{R}Nds_C, \tag{13.33}$$

where ds_C is the elemental displacement of the centre C of the wheel.

If N=*const*, then the total work done by the forces resisting rolling will be

$$A_{roll} = -kN\varphi_1 = -\frac{k}{R}Ns_C. \tag{13.34}$$

As the quantity k/R is small, rolling friction can, in the first approximation, be neglected as compared with other resisting forces.

7. Theorem of the change in the kinetic energy of a system

The theorem proved in §4 is valid for any point of a system. Therefore, if we take any particle of mass m_k and velocity υ_k belonging to a system, we have for this particle

$$d\left(\frac{m_k \upsilon_k^2}{2}\right) = dA_k^e + dA_k^i,$$

where $dA^e{}_k$ and $dA^i{}_k$ are the elementary work done by the external and internal forces acting on the particle.

If we write similar equations for all the particles of a system and add them, we obtain:

$$d\left(\sum_k \frac{m_k \upsilon_k^2}{2}\right) = \sum_k dA_k^e + \sum_k dA_k^i,$$

or

$$dT = \sum_k dA_k^e + \sum_k dA_k^i. \tag{13.35}$$

Equation (13.35) states the **theorem of the change in the kinetic energy of a system in differential form**. Integrating both parts of the equation in the limits corresponding to the displacement of the system from some initial position where the kinetic energy is T_0 to a position where it is T_1, we obtain:

$$T_1 - T_0 = \sum_k A_k^e + \sum_k A_k^i. \tag{13.36}$$

This equation states the **theorem of the change in kinetic energy in final form**: *The change in the kinetic energy of a system during any displacement is equal to the sum of the work done by all the external and internal forces acting on the system in that displacement*.

(1) Non-deformable systems. A non-deformable system is defined as one in which the distance between the points of application of the internal forces does not change during the motion of the system. Special cases of such systems are a rigid body and an inextensible string.

Let two points B_1 and B_2 of a non-deformable system (Fig. 13.6) be acting on each other with forces $\mathbf{F}^i_{12}$ and $\mathbf{F}^i_{21}$ ($\mathbf{F}^i_{12}=-\mathbf{F}^i_{21}$ and let their velocities at some instant be $\boldsymbol{\upsilon}_1$ and $\boldsymbol{\upsilon}_2$. Their displacements in a time interval dt will be $ds_1=\upsilon_1 dt$ and $ds_1=\upsilon_1 dt$ directed along vectors $\boldsymbol{\upsilon}_1$ and $\boldsymbol{\upsilon}_2$. But as line B_1B_2 is non-deformable, it follows from the laws of kinematics that the projections of vectors $\boldsymbol{\upsilon}_1$ and $\boldsymbol{\upsilon}_2$, and consequently of the displacements ds_1 and ds_2, on the direction of B_1B_2 will be equal, i.e., $B_1B'_1=B_2B'_2$. Then the elemental work done by forces $\mathbf{F}^i_{12}$ and $\mathbf{F}^i_{21}$ will be equal in magnitude and opposite in sense, and their sum will be zero. This holds for all internal forces in any displacement of a system.

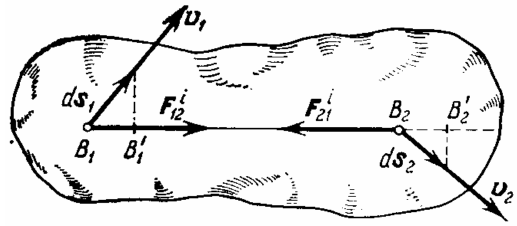


Fig. 13.6

We conclude from this that the sum of the work done by all the internal forces of a non-deformable system is zero, and Eqs. (13.35) or (13.36) take the form

$$dT = \sum_k dA_k^e \quad \text{or} \quad T_1 - T_0 = \sum_k A_k^e . \tag{13.37}$$

(2) Systems with ideal constraints. Consider a system with con¬straints that do not change with time. Dividing all the external and internal forces acting on the particles of the system into *active forces* and the *reactions of the constraints*, Eq. (13.35) can be written in the form:

$$dT = \sum_k dA_k^a + \sum_k dA_k^r ,$$

where dA^a_k is the elementary work done by the external and internal forces acting on the k-th particle of the system, and dA^r_k is the elementary work done by the reactions of the external and internal constraints acting on that particle.

We see that the change in the kinetic energy of the system depends on the work done by both the acting forces and the reactions of the constraints. However, we can introduce the concept of "ideal" mechanical systems in which constraints do not affect the change in kinetic energy in the motion of the system. Such constraints should, evidently, satisfy the condition:

$$\sum_k dA_k^r = 0. \tag{13.38}$$

If for constraints that do not change with time the sum of the work done by all the reactions in an elementary displacement of a system is zero, such constraints are called *ideal*. Here are some known examples of ideal constraints.

It was established that if a constraint is a fixed smooth surface (or curve), for which friction can be neglected, the work done by the reaction **N** in the motion of a body along that surface (curve) is zero. Then, it was shown that, neglecting deformation, if a body rolls without slipping on a rough surface, the work done by the normal reaction **N** and the force of friction **F** (i.e., the tangential component of the reaction) is zero. Also, the work done by the reaction **R** of a hinge is, neglecting friction, zero, as in any displacement of the system the point of application of force **R** is fixed. Finally, if the material particles B_1 and B_2 in Fig. 13.6 are assumed to be connected by a rigid rod B_1B_2, the forces $\mathbf{F}^i{}_{12}$ and $\mathbf{F}^i{}_{21}$ will be the reactions of the constraint; the work of each of these reactions in the displacement of the system is not zero, but their sum, as shown, is zero. Thus, all the mentioned constraints can, with the assumptions made, be regarded as ideal.

In the case of a mechanical system subject solely to ideal constraints that do not change with time we obviously have:

$$dT = \sum_k dA_k^a \quad \text{or} \quad T_1 - T_0 = \sum_k A_k^a. \tag{13.39}$$

Thus, the change in the kinetic energy of a system with ideal constraints that do not change with time is, in any displacement, equal to the sum of the work done in that displacement by the active external and internal forces.

All the foregoing theorems made it possible to exclude the internal forces from the equations of motion, but all the external forces, including the immediately unknown reactions of the external constraints, entered the equations. The theorem of the change in kinetic energy is useful because in the case of ideal constraints that do not change with time it makes it possible to exclude *all* the immediately unknown reactions of the constraints from the equations of motion.

8. Conservative force field and force function

The work done in a displacement M_1M_2 by a force **F** applied at a point M of a body is computed according to Eq. (13.7):

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} dA = \int_{(M_0)}^{(M_1)} (F_x dx + F_y dy + F_z dz). \quad (13.40)$$

As pointed out in §2, the integral on the right can be evaluated without knowledge of the law of motion involved (i.e., of the dependence of x, y, z on time) only if the force depends solely on the location of the point, i.e., on its x, y, z coordinates. Such forces are said to form a force field, or field of force. *A force field is defined as a region of space in which any article experiences a force of certain magnitude and direction*. Examples are planetary or stellar gravitational fields. As any force can be defined by its projections on a set of coordinate axes, a force field can be described by the equations:

$$F_x = \Phi_1(x,y,z), \quad F_y = \Phi_2(x,y,z), \quad F_z = \Phi_3(x,y,z). \quad (13.41)$$

But in the most general case, to compute the work done by such forces, in Eq. (13.40) it is necessary to go over to one variable in the integrand; for example, one must know the dependencies $y=f_1(x)$ and $z=f_2(x)$, which give the spatial equation of the curve that is the path of particle M. Consequently, in the most general case the work done by the forces constituting a force field depends on the type of path of the point of application of the relevant force.

However, if the integrand in Eq. (13.40), which represents the elementary work done by force F, is the full differential of a function $U(x,y,z)$, i.e., if

$$dA = dU(x,y,z), \quad \text{or} \quad F_x dx + F_y dy + F_z dz = dU(x,y,z), \quad (13.42)$$

the work A can be computed without knowing the path of point M.

The function U of the coordinates x, y, z, the differential of which equals the elementary work, is called a *force function*. A force field for which there is a force function is called a *conservative force field*, and the forces acting in that field are called *conservative forces*. We shall regard a force function as a unique function of coordinates.

Substituting the expression for dA from Eq. (13.42) into Eq. (13.40), we obtain:

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} dU(x, y, z) = U_2 - U_1, \tag{13.43}$$

where $U_1=U(x_1,y_1,z_1)$ and $U_2=U(x_2,y_2,z_2)$ are the values of the force functions at points M_1 and M_2 of the field, respectively. Consequently, the work done by a conservative force acting on a moving particle equals the difference between the values of the force function at the terminal and initial points of the displacement and does not depend on the particle's path. In a displacement along a closed path $U_2=U_1$, and the work done by a conservative force is zero.

The basic property of a conservative force field is that the work done by its forces acting on a moving material particle depends only on the particle's initial and final positions and does not depend on its path followed or the law of motion.

When the work done by a force depends on the path or law of motion of the point at which it is applied, the force is said to be *nonconservative*, or *dissipative*. Examples are friction and the resistance of a medium.

If the relationship (13.42) is found to apply, the force function can be determined from the equation

$$U = \int dA + C, \quad \text{or} \quad U = \int (F_x dx + F_y dy + F_z dz) + C, \tag{13.44}$$

where C is a constant having any value [it is apparent from Eq. (13.43) that work does not depend on C]. However, it is conventionally assumed that at some point 0, called the "zero point", $U_O=0$, and C is determined on that basis.

9. Potential energy

For conservative forces we can introduce the concept of *potential energy* as a measure of the capacity of a particle for doing work by virtue of its position in the force field. In order to compare different "capacities for doing work", we must agree on the choice of a zero point 0, in which we assume the capacity to do work to be zero (the choice of the zero point, as of any initial point or origin, is arbitrary). *The potential energy of a particle in any configuration M is defined as the scalar quantity V equal to* *the work done on the particle by the forces of a field in the passage from configuration M to the zero configuration*:

$$V = A_{(MO)} .$$

It follows from the definition that potential energy is dependent on the coordinates of the particle M, i.e., $V=V(x,y,z)$.

Assuming that the zero points of the functions $V(x,y,z)$ and $U(x,y,z)$ coincide, we have $U_0=0$ and, by Eq. (13.43), $A_{(MO)}=U_0-U=-U$, where U is the force function at point M of the field; whence,

$$V(x, y, z) = -U(x, y, z) ,$$

i.e., the potential energy at any point of a force field is equal to the magnitude of the force function at that point taken with the opposite sign.

It is thus apparent that in investigating the properties of a conservative force field we can replace the force function with potential energy. In particular, in computing the work done by a conservative force we can use instead of Eq. (13.43) the formula

$$A_{(M_0 M_1)} = V_1 - V_2 , \tag{13.45}$$

Thus, the work done by a conservative force is equal to the difference between a moving particle's potential energy in its initial and final positions.

10. The law of conservation of mechanical energy

Let us assume that all the external and internal forces acting on a system are conservative forces. Then, for any particle belonging to the system, the work done by the applied forces is

$$A_k = V_{k0} - V_{k1} ,$$

and for all the external and internal forces

$$\sum_k A_k = \sum_k V_{k0} - \sum_k V_{k1} = V_0 - V_1 ,$$

where $V_{0(1)}$ is the potential energy of the whole system.

Substituting this expression for work into Eq. (13.36), we obtain:

$$T_1 - T_0 = V_0 - V_1 \quad \text{or} \quad T_1 + V_1 = V_0 + T_0 = const .$$

Thus, in the motion of a system subjected to the action of only conservative forces, the sum of the kinetic and potential energies of the system remains constant for any configuration. This is the *law of conservation of mechanical energy*, which is a particular case of the general physical law of conservation of energy. The quantity $T+V$ is called the *total mechanical energy of the system*.

If the acting forces include dissipative forces, such as friction, the total mechanical energy of the system will decrease during its motion due to transformation into other forms of energy, e.g., thermal energy.

The whole meaning of the foregoing law becomes apparent when it is considered in connection with the general physical law of conservation of energy. However, in solving purely mechanical problems, the theorem of the change in the kinetic energy of a system can always be immediately applied

LECTURE 15
THEOREM OF THE MOTION OF THE CENTER OF MASS OF A SYSTEM
1. The differential equations of motion of a system

Suppose we have a system of n particles. Choosing any particle of mass m_k, belonging to the system, let us denote the resultant of all the external forces acting on the particle (both active forces and the forces of reaction) by the symbol $\mathbf{F}^{(e)}{}_k$, and the resultant of all the internal forces by $\mathbf{F}^{(e)}{}_i$. If the particle has an acceleration $\boldsymbol{a}_k$, then, by the fundamental law of dynamics,

$$m_k \boldsymbol{a}_k = \mathbf{F}_k^{(e)} + \mathbf{F}_k^{(i)}.$$

Similar results are obtained for any other particle, whence, for the whole system, we have:

$$\left.\begin{aligned} m_1 \boldsymbol{a}_1 &= \mathbf{F}_1^{(e)} + \mathbf{F}_1^{(i)}, \\ m_2 \boldsymbol{a}_2 &= \mathbf{F}_2^{(e)} + \mathbf{F}_2^{(i)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ m_n \boldsymbol{a}_n &= \mathbf{F}_n^{(e)} + \mathbf{F}_n^{(i)}. \end{aligned}\right\} \qquad (15.1)$$

These equations, from which we can develop the law of motion of any particle of the system, are called the *differential equations of motion of a system in vector form.* In the most general case the forces in the right side of the equations depend on time, coordinates of the particles of the system, and velocities.

By projecting Eqs. (15.1) on coordinate axes, we can obtain the differential equations of motion of a given system in terms of the projections on these axes.

The complete solution of the principal problem of dynamics for a system would be to develop the equation of motion for each particle of the system from the given forces by integrating the corresponding differential equations. For two reasons, however, this solution is not usually employed. Firstly, the solution is too involved and will almost inevitably lead into insurmountable mathematical difficulties. Secondly, in solving problems of mechanics it is usually sufficient to know certain overall characteristics of the motion of a system, without investigating the motion of each particle. These overall characteristics can be found with the help of the *general theorems of system dynamics,* which we shall now study.

The main application of Eqs. (15.1) or their corollaries will be to develop the respective general theorems.

2. Theorem of motion of centre of mass

In many cases the nature of the motion of a system (especially of a rigid body) is completely described by the law of motion of its centre of mass. To develop this law, let us take the equations of motion of a system (15.1) and add separately their left and right sides. We obtain:

$$\sum_k m_k \boldsymbol{a}_k = \sum_k \mathbf{F}_k^{(e)} + \sum_k \mathbf{F}_k^{(i)} . \qquad (15.2)$$

Let us transform the left side of the equation. For the radius vector of the centre of mass we have, from Eq. (12.2),

$$\sum_k m_k \mathbf{r}_k = M\mathbf{r}_C .$$

Taking the second derivative of both sides of this equation with respect to time, and noting that the derivative of a sum equals the sum of the derivatives, we find that

$$\sum_k m_k \frac{d^2\mathbf{r}_k}{dt^2} = M\frac{d^2\mathbf{r}_C}{dt^2} ,$$

or

$$\sum_k m_k \boldsymbol{a}_k = M\boldsymbol{a}_C , \qquad (15.3)$$

where $\boldsymbol{a}_C$ is the acceleration of the centre of mass of the system. As the internal forces of a system is equal zero by substituting all the developed expressions into Eq. (15.2), we obtain finally:

$$\sum_k m_k \boldsymbol{a}_k = \sum_k \mathbf{F}_k^{(e)} . \tag{15.4}$$

Eq. (15.4) states the theorem of the motion of the centre of mass of a system. Its form coincides with that of the equation of motion of a particle of mass $m = M$ where the acting forces are equal to $\mathbf{F}^{(e)}{}_k$. We can therefore formulate the **theorem of the motion of the centre of mass** as follows: *The centre of mass of a system moves as if it were a particle of mass equal to the mass of the whole system to which are applied all the external forces acting on the system.*

Projecting both sides of Eq. (15.4) on the coordinate axes, we obtain:

$$M\frac{d^2 x_C}{dt^2} = \sum_k F_{kx}^{(e)}, \quad M\frac{d^2 y_C}{dt^2} = \sum_k F_{ky}^{(e)}, \quad M\frac{d^2 z_C}{dt^2} = \sum_k F_{kz}^{(e)} . \tag{15.5}$$

These are the *differential equations of motion of the centre of mass* in terms of the projections on the coordinate axes.

The theorem is valuable for the following reasons:

(1) It justifies the use of the methods of particle dynamics. It follows from Eqs. (15.5) that the solutions developed on the assumption that a given body is equivalent to a particle define the law of motion of the centre of mass of that body. Thus, these solutions have a concrete meaning.

In particular, if a body is in translational motion, its motion is completely specified by the motion of its centre of mass, and consequently, a body in translatory motion can always be treated as a particle of mass equal to the mass of the body. In all other cases, a body can be treated as a particle only when the position of its centre of mass is sufficient to specify the position of the body

(2) The theorem makes it possible, in developing the equation of motion for the centre of mass of any system, to ignore all unknown internal forces. This is of special practical value.

Поскольку механическая система, это прежде всего совокупность материальных точек, то тогда количество движения системы точек – сумма количеств движения отдельных ее частей

3. The law of conservation of motion of centre of mass

The following important corollaries arise from the theorem of the motion of centre of mass:

(1) Let the sum of the external forces acting on a system be zero:

$$\sum_k \mathbf{F}_k^{(e)} = 0 .$$

It follows, then, from Eq. (15.4) that $\boldsymbol{a}_C = 0$ or $\boldsymbol{\upsilon}_C = \text{const.}$

Thus, if the sum of all the external forces acting on a system is zero, the centre of mass of that system moves with velocity of constant magnitude and direction, i.e., uniformly and rectilinearly. In particular, if the centre of mass was initially at rest, it will remain at rest. The action of the internal forces, we see, does not affect the motion of the centre of mass.

(2) Let the sum of the external forces acting on a system be other than zero, but let the sum of their projections on one of the coordinate axes (axis *x,* for instance) be zero:

$$\sum_k F_{kx}^{(e)} = 0 .$$

The first of Eqs. (15.5) then gives

$$\frac{d^2 x_C}{dt^2} = 0 \quad \text{or} \quad \frac{dx_C}{dt} = \upsilon_{Cx} = \text{const} .$$

Thus, if the sum of the projections on an axis of all the external forces acting on a system is zero, the projection of the velocity of the centre of mass of the system on that axis is a constant quantity. In particular, if at the initial moment $\upsilon_{Cx} = 0$, it will remain zero at any subsequent instant, i.e., the centre of mass of the system will not move along axis x ($x_C = \text{const.}$).

The above results express the *law of conservation of motion of the centre of mass of a system.*

4. Linear momentum of a system

The linear momentum, or simply the momentum, of a system is defined as the vector quantity **Q** *equal to the geometric sum* (*the principal vector*) *of the momenta of all the particles of the system*:

$$\mathbf{Q} = \sum_k m_k \boldsymbol{\upsilon}_k . \tag{15.6}$$

It can be seen from the diagram that, irrespective of the velocities of the particles (provided that they are not parallel) the momentum vector can take any value, or even be zero when the polygon constructed with the vectors $m_k\mathbf{v}_k$ as its sides is closed. Consequently, the quantity **Q** does not characterise the motion of the system completely.

Let us develop a formula with which it is much more convenient to compute **Q** and also to explain its meaning. It follows from Eq. (12.2) that

$$\sum_k m_k \mathbf{r}_k = M\mathbf{r}_C .$$

Differentiating both sides with respect to time, we obtain:

$$\sum_k m_k \frac{d\mathbf{r}_k}{dt} = M\frac{d\mathbf{r}_C}{dt} \quad \text{or} \quad \sum_k m_k \mathbf{v}_k = M\mathbf{v}_C .$$

whence we find that

$$\mathbf{Q} = M\mathbf{v}_C . \tag{15.7}$$

i.e., *the momentum of a system is equal to the product of the mass of the whole system and the velocity of its centre of mass*. This equation is especially convenient in computing the momentum of rigid bodies.

It follows from Eq. (15.7) that if the motion of a body (or a system) is such that the centre of mass remains motionless, the momentum of the body is zero. Thus, the momentum of a body rotating about a fixed axis through its centre of mass is zero.

If, on the other hand, a body is in relative motion, the quantity **Q** will not characterise the rotational component of the motion about the centre of mass. Thus, for a rolling wheel, $\mathbf{Q} = M\mathbf{v}_C$, regardless of how the wheel rotates about its centre of mass *C*.

We see, therefore, that momentum characterises only the translatory motion of a system, which is why it is often called *linear momentum.*

5. Linear momentum of a system

Consider a system of *n* particles. Writing the differential equations of motion (15.1) for this system and adding them, we obtain:

$$\sum_k m_k \boldsymbol{a}_k = \sum_k \mathbf{F}_k^{(e)} + \sum_k \mathbf{F}_k^{(i)} .$$

From the property of internal forces the last summation is zero. Furthermore,

$$\sum_k m_k \boldsymbol{a}_k = \frac{d}{dt}\sum_k m_k \mathbf{v}_k = \frac{d\mathbf{Q}}{dt},$$

and we finally have

$$\frac{d\mathbf{Q}}{dt} = \sum_k \mathbf{F}_k^{(e)}. \tag{15.8}$$

Eq. (15.8) states the **theorem of the change in the linear momentum of** a **system in differential form*:*** *The derivative of the linear momentum of a system with respect to time is equal to the geometrical sum of all the external forces acting on the system.*

In terms of projections on cartesian axes we have

$$\frac{dQ_x}{dt} = \sum_k F_{kx}^{(e)}, \quad \frac{dQ_y}{dt} = \sum_k F_{ky}^{(e)}, \quad \frac{dQ_z}{dt} = \sum_k F_{kz}^{(e)}. \tag{15.9}$$

Let us develop another expression for the theorem. Let the momentum of a system be $\mathbf{Q}_0$ at time $t = 0$, and at time t_1 let it be $\mathbf{Q}_1$.

Multiplying both sides of Eq. (15.8) by dt and integrating, we obtain:

$$\mathbf{Q}_1 - \mathbf{Q}_0 = \sum_k \int_0^{t_1} \mathbf{F}_k^{(e)} dt,$$

or

$$\mathbf{Q}_1 - \mathbf{Q}_0 = \sum_k \mathbf{S}_k^{(e)}, \tag{15.10}$$

as the integrals to the right give the impulses of the external forces.

Eq. (15.10) states the **theorem of the change in the linear momentum of a system in integral form:** *The change in the linear momentum of a system during any time interval is equal to the sum of the impulses of the external forces acting on the body during the same interval of time.*

In terms of projections on cartesian axes we have

$$\left.\begin{aligned} Q_{1x} - Q_{0x} &= \sum_k S_{kx}^{(e)}, \\ Q_{1y} - Q_{0y} &= \sum_k S_{ky}^{(e)}, \\ Q_{1z} - Q_{0z} &= \sum_k S_{kz}^{(e)}. \end{aligned}\right\}, \tag{15.11}$$

Consequently, the theorem of the motion of centre of mass and the theorem of the change in the momentum of a system are, in effect, two forms of the same theorem. Whenever the motion of a rigid body (or system of bodies) is being investigated, both theorems may be used, though Eq. (15.7) is usually more convenient.

For a continuous medium (a fluid), however, the concept of centre of mass of the whole system is virtually meaningless, and the theorem of the change in the momentum of a system is used in the solution of such problems. This theorem is also very useful in investigating the theory of impact and jet propulsion.

The practical value of the theorem is that it enables us to exclude from consideration the immediately unknown internal forces (for instance, the reciprocal forces acting between the particles of a liquid).

6. The law of conservation of linear momentum

The following important corollaries arise from the theorem of the change in the momentum of a system:

(1) Let the sum of all the external forces acting on a system be zero:

$$\sum_k \mathbf{F}_k^{(e)} = 0 .$$

It follows from Eq. (15.8) that in this case $\mathbf{Q}$ = const. Thus, if the sum of all the external forces acting on a system is zero, the momentum vector of the system is constant in magnitude and direction.

(2) Let the external forces acting on a system be such that the sum of their projections on any axis Ox is zero:

$$\sum_k F_{kx}^{(e)} = 0 .$$

It follows from Eqs. (15.9) that in this case Q_x = const. Thus, if the sum of the projections on any axis of all the external forces acting on a system is zero, the projection of the momentum of that system on that axis is a constant quantity.

These results express the *law of conservation of the linear momentum of a system.* It follows from the above that internal forces are incapable of changing the total momentum of a system.

LECTURE 16

THEOREM OF THE CHANGE IN THE ANGULAR MOMENTUM OF A SYSTEM

1. Theorem of the change in the angular momentum of a particle (the principle of moments)

Often, in analysing the motion of a particle, it is necessary to consider the change not in the vector $m\boldsymbol{\upsilon}$ itself but in its moment. The moment of the vector $m\boldsymbol{\upsilon}$ with respect to any centre O or axis z is denoted by the symbol $\boldsymbol{m}_0(m\boldsymbol{\upsilon})$ or $\boldsymbol{m}_z(m\boldsymbol{\upsilon})$ and is called the *moment of momentum.* or *angular momentum* with respect to that centre or axis. The moment of vector $m\boldsymbol{\upsilon}$ is calculated in the same way as the moment of a force. Vector $m\boldsymbol{\upsilon}$ is considered to be applied to the moving particle. In magnitude $|\boldsymbol{m}_0(m\boldsymbol{\upsilon})|=m\upsilon h$, where h is the perpendicular distance from O to the position line of the vector $m\upsilon$ (see Fig. 16.1).

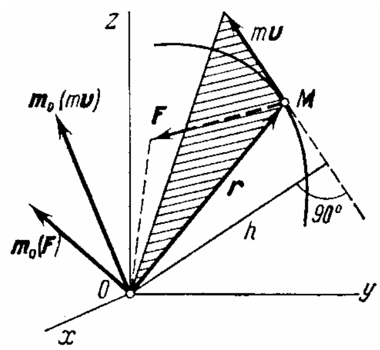


Fig. 16.1

(1) Principle of moments about an axis. Consider a particle of mass m moving under the action of a force **F**. Let us establish the dependence between the moments of the vectors $m\boldsymbol{\upsilon}$ and **F** with respect to any fixed axis z:

$$m_z(\mathbf{F}) = xF_y - yF_x. \tag{16.1}$$

Similarly, form $m_z(m\boldsymbol{\upsilon})$, and taking m out of the parentheses, we have:

$$m_z(m\boldsymbol{\upsilon}) = m(x\upsilon_y - y\upsilon_x). \tag{16.2}$$

Differentiating this equation with respect to time, we obtain:

$$\frac{d}{dt}\left[m_z(m\boldsymbol{\upsilon})\right] = m\left(\frac{dx}{dt}\upsilon_y - \frac{dy}{dt}\upsilon_x\right) + \left(mx\frac{d\upsilon_y}{dt} - my\frac{d\upsilon_x}{dt}\right).$$

The first member in the right-hand side of the equation is zero. From Eq. (16.1), the second member is equal to $m_z(\mathbf{F})$, since, from the fundamental law of dynamics,

$$m\frac{d\upsilon_y}{dt} = F_y, \quad m\frac{d\upsilon_x}{dt} = F_x.$$

Finally, we have:

$$\frac{d}{dt}[m_z(m\mathbf{v})] = m_z(\mathbf{F}). \qquad (16.3)$$

This equation states the **principle of moments about an axis**: *The derivative of the angular momentum oj a particle about any axis with respect to time is equal to the moment of the acting force about the same axis*.

From Eq. (16.3) it follows that if $m_z(\mathbf{F})=0$, then $m_z(m\mathbf{v})=\text{const}$, i.e. if the moment of the acting force about an axis is zero, the angular momentum of this particle about this axis is constant in magnitude and direction.

(2) Principle of moments about a centre. Let us find for a particle moving under the action of a force **F** (Fig. 16.1) the relation between the moments of vectors $m\mathbf{v}$ and **F** with respect to any fixed centre *O*. It was shown early

$$\mathbf{m}_O(\mathbf{F}) = \mathbf{r} \times \mathbf{F}.$$

Similarly,

$$\mathbf{m}_O(m\mathbf{v}) = \mathbf{r} \times m\mathbf{v}.$$

Vector $\boldsymbol{m}_0(\mathbf{F})$ is normal to the plane through *O* and vector **F**, while vector $\boldsymbol{m}_0(m\mathbf{v})$ is normal to the plane through *O* and vector $m\mathbf{v}$. Differentiating the expression $\boldsymbol{m}_0(m\mathbf{v})$ with respect to time, we obtain:

$$\frac{d}{dt}[\mathbf{r} \times m\mathbf{v}] = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m\frac{d\mathbf{v}}{dt}\right) = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\boldsymbol{a}).$$

But $\mathbf{v} \times m\mathbf{v} = 0$, as the vector product of two parallel vectors, and $m\boldsymbol{a} = \mathbf{F}$. Hence,

$$\frac{d}{dt}[\mathbf{r} \times m\mathbf{v}] = \mathbf{r} \times \mathbf{F}, \qquad (16.4)$$

or

$$\frac{d}{dt}[\mathbf{r} \times m\mathbf{v}] = \mathbf{m}_O(\mathbf{F}). \qquad (16.5)$$

This is the **principle of moments about a centre:** *The derivative of the angular momentum of a particle about any fixed centre with respect to time is equal to the moment of the force acting on the particle about the same centre.* An analogous theorem is true for the moments of vector $m\mathbf{v}$ and force **F** with respect to any axis *z*, which is evident if we project both sides of Eq. (16.5) on that axis. This was proved directly in item (1). The mathematical statement of the theorem of moments about an axis is given in Eq. (16.3) above.

From Eq. (16.5) it follows that if $\mathbf{m}_0(\mathbf{F})=0$, then $\mathbf{m}_0(m\mathbf{v})=\text{const.}$, i.e., if the moment of the acting force relative to a centre is zero, the angular momentum of this particle about the same centre is constant in magnitude and direction. This result is of great importance in the case of motion under the action of a *central force*.

2. Total angular momentum of a system

The total angular momentum of a system with respect to a centre O is defined as the quantity $\mathbf{K}_O$ *equal to the geometrical sum of the angular momenta of all the particles of the system with respect to that centre.*

$$\mathbf{K}_O = \sum_k \mathbf{m}_O(m_k \mathbf{v}_k). \tag{16.6}$$

The angular momenta of a system with respect to each of three rectangular coordinate axes are found similarly:

$$K_x = \sum_k m_x(m_k \mathbf{v}_k), \quad K_y = \sum_k m_y(m_k \mathbf{v}_k), \quad K_z = \sum_k m_z(m_k \mathbf{v}_k). \tag{16.7}$$

K_x, K_y, and K_z are the respective projections of vector $\mathbf{K}_O$ on the coordinate axes.

Just as the momentum of a system is a characteristic of its translational motion, the *total angular momentum of a system* is a characteristic of its rotational motion.

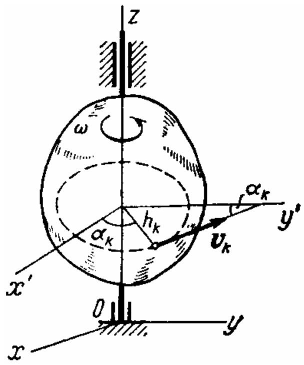


Fig. 16.2

To understand the physical meaning of $\mathbf{K}_O$ and obtain the formulas necessary for problem solutions, let us compute the angular momentum of a body rotating about a fixed axis (Fig. 16.2). As usual, we shall determine vector $\mathbf{K}_O$ in terms of its projections K_x, K_y, and K_z.

First, let us find the formula for determining $\mathbf{K}_z$, i.e., the angular momentum of a rotating body with respect to the axis of rotation.

The linear velocity of any particle of the body at a distance h_k from the axis is ωh_k. Consequently, for that particle $m_z(m_k\mathbf{v}_k)=m_k v_k h_k = m_k \omega h_k^2$. Then, taking the common multiplier ω outside of the parentheses, we obtain for the whole body:

$$K_z = \sum_k m_z(m_k \mathbf{v}_k) = \left(\sum_k m_k h_k^2 \right) \omega .$$

The quantity in the parentheses is the moment of inertia of the body with respect to axis z. We finally obtain:

$$K_z = J_z \omega . \tag{16.8}$$

Thus, *the angular momentum of a rotating body with respect to the axis of rotation is equal to the product of the moment of inertia of the body and its angular velocity.*

If a system consists of several bodies rotating about the same axis, then, apparently,

$$K_z = J_{1z}\omega_1 + J_{2z}\omega_2 + \ldots J_{nz}\omega_n . \tag{16.9}$$

The analogy between linear momentum of a system and angular momentum will be readily noticed: the momentum of a body is the product of its mass (the quantity characterising the body's inertia in translational motion) and its velocity; the angular momentum of a body is equal to the product of its moment of inertia (the quantity characterising a body's inertia in rotational motion) and its angular velocity.

Let us now compute the quantities K_x and K_y. As in the determination of the moment of a force, to determine $m_x(m_k\mathbf{v}_k)$ we must project vector $m_k\mathbf{v}_k$ on plane Oyz, i.e., on axis y', and find the moment of the projection with respect to point O. We obtain $m_x(m_k\mathbf{v}_k) = -(m_k \upsilon_k \cos\alpha_k) z_k$. But $\upsilon_k \cos\alpha_k = \omega h_k \cos\alpha_k = \omega x_k$ as from Fig. 16.2 it is apparent that $h_k \cos\alpha_k = x_k$. Consequently, taking the common multiplier outside of the parentheses, we find that

$$K_x = \sum_k m_x(m_k \mathbf{v}_k) = -\left(\sum_k m_k x_k z_k \right) \omega .$$

The sum in the parentheses is the product of inertia J_{xz}. A similar expression is obtained for K_y, with y_k substituted for x_k. Finally, we obtain:

$$K_x = -J_{xz}\omega, \quad K_y = -J_{yz}\omega . \tag{16.10}$$

Thus, the angular momentum of a rotating body with respect to a centre O on the axis of rotation Oz is a vector $\mathbf{K}_O$ whose projections on the x, y, z axes are given by the formulas (16.8) and (16.10). It will be observed that in the most general case vector $\mathbf{K}_O$ is not directed along the axis of rotation Oz. But if the axis of rotation is, for point O, the principal axis of inertia of the body (in particular, the axis of symmetry), then $J_{xz} = J_{yz} = 0$, and

K_x=K_y=0 and K_O=K_z. Consequently, if a body rotates about an axis that is its principal axis of inertia with respect to point O (or about its axis of symmetry), then vector $\mathbf{K}_O$ is directed along the axis of rotation and is equal in magnitude to K_z, i.e., to $J_z\omega$.

3. Theorem of the change in the total angular momentum of a system (the principle of moments)

The principle of moments, which was proved for a single particle, is valid for all the particles of a system. If, therefore, we consider a particle of mass m_k and velocity $\mathbf{v}_k$ belonging to a system, we have for that particle:

$$\frac{d}{dt}\left[\mathbf{m}_O(m_k\mathbf{v}_k)\right]=\mathbf{m}_O(\mathbf{F}_k^e)+\mathbf{m}_O(\mathbf{F}_k^i),$$

where F_k^e and F_k^i are the resultants of all the external and internal forces acting on the particle.

Writing such equations for all the particles of the system and adding them, we obtain:

$$\frac{d}{dt}\left[\sum_k \mathbf{m}_O(m_k\mathbf{v}_k)\right]=\sum_k \mathbf{m}_O(\mathbf{F}_k^e)+\sum_k \mathbf{m}_O(\mathbf{F}_k^i).$$

But from the properties of the internal forces of a system, the last summation vanishes. Hence, taking into account Eq. (16.6), we obtain finally:

$$\frac{d\mathbf{K}_O}{dt}=\sum_k \mathbf{m}_O(\mathbf{F}_k^e). \tag{16.11}$$

This equation states the following **principle of moments** for a system: *The derivative of the total angular momentum of a system about any fixed centre with respect to time is equal to the sum of the moments of all the external forces acting on that system about that centre*.

Projecting both sides of Eq. (16.11) on a set of fixed axes $Oxyz$ we obtain:

$$\frac{dK_x}{dt}=\sum_k m_x(\mathbf{F}_k^e),\quad \frac{dK_y}{dt}=\sum_k m_y(\mathbf{F}_k^e),\quad \frac{dK_z}{dt}=\sum_k m_z(\mathbf{F}_k^e). \tag{16.12}$$

Equations (16.12) express the principle of moments with respect to any fixed axis.

The theorem just proved is widely used in studying the rotation of a body about a fixed axis, and also in the theory of gyroscopic motion and the theory of impact. This, however, is not all. It was proved in the course of kinematics that the most general motion of a body is a combination of a translation together with some pole and a rotation about that pole. If the pole is located in the centre of mass, the translational component of the motion can be investigated by applying the theorem of the motion of the centre of mass, and the rotational component, by the theorem of moments. This indicates the theorem's importance in studying the motion of free bodies and, in particular, in studying plane motion.

The principle of moments is also convenient in investigating the rotation of a system, because, analogous to the theorem of the change in linear momentum, it makes it possible to exclude from consideration all immediately unknown internal forces.

Theorem of Moments With Respect to a Centre of Mass: *For axes in translational motion together with the centre of mass of a system, the theorem of moments with respect to the centre of mass has the same form as with respect to a fixed, centre.*

$$\frac{d\mathbf{K}_C}{dt} = \sum_k \mathbf{m}_C(\mathbf{F}_k^e)\,. \qquad (16.13)$$

4. The law of conservation of the total angular momentum

The following important corollaries can be derived from the principle of moments.

(1) Let the sum of the moments of all the external forces acting on a system with respect to a centre O be zero:

$$\sum_k \mathbf{m}_O(\mathbf{F}_k^e) = 0\,.$$

It follows, then, from Eq. (16.11) that $\mathbf{K}_O$=const. Thus, *if the sum of the moments of all external forces acting on a system taken with respect to any centre is zero, the total angular momentum of the system with respect to that centre is constant in magnitude and direction.*

(2) Let the external forces acting on a system be such that the sum of their moments with respect to any fixed axis Oz is zero:

$$\sum_k m_z(\mathbf{F}_k^e) = 0\,.$$

It follows, then, from Eqs. (16.12) that K_z=const. Thus, *if the sum of the moments of all the external forces acting on a system with respect to any axis is zero, the total angular momentum of the system with respect to that axis is constant.*

These conclusions express the *law of conservation of the total angular momentum of a system.* It follows from them that internal forces cannot change the total angular momentum of a system.

Rotating Systems. Consider a system rotating about an axis *Oz* which is fixed or passes through the centre of mass. By Eq. (16.8), $K_z=J_z\omega$, and if

$$\sum_k \mathbf{m}_O(\mathbf{F}_k^e) = 0.$$

then

$$J_z\omega = \text{const}.$$

This leads us to the following conclusions:

(a) If a system is *non-dejormable* (a rigid body), then J_z=const, whence ω=const. That is, a rigid body will rotate about a fixed axis with a constant angular velocity.

(b) If a system is *deformable,* it will have particles which, under the action of internal (or external) forces, may move away from the axis, thereby increasing J_z, or approach the axis, thereby decreasing J_z. But as $J_z\omega$=const, the angular velocity of the system will decrease as the moment of inertia increases, and increase as the moment of inertia decreases. Thus, the action of internal forces can change the angular velocity of a rotating system, as the constancy of K_z does not, in the general case, mean the constancy of ω.

5. Rotation of a rigid body

Let there be a system of forces $\mathbf{F}_1^e$, $\mathbf{F}_2^e$, ..., $\mathbf{F}_n^e$ acting on a rigid body with a fixed axis of rotation *z*. Also acting on the body are the reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ of the bearings. As the moments of forces $\mathbf{R}_A$ and $\mathbf{R}_B$ with respect to the axis are zero, we have

$$\frac{dK_z}{dt} = M_z^e,$$

where

$$M_z^e = \sum_k m_z(\mathbf{F}_k^e).$$

We shall call the quantity M_z^e the *turning moment*, or *torque*.

Substituting the expression $K_z = J_z\omega$ into the foregoing equation, we obtain:

$$J_z \frac{d\omega}{dt} = M_z^e \quad \text{or} \quad J_z \frac{d^2\varphi}{dt^2} = M_z^e. \tag{16.14}$$

Eq. (16.14) is the differential equation of the rotational motion of a rigid body. It follows from the equation that the product of the moment of inertia of a body with respect to its axis of rotation and its angular acceleration is equal to the turning moment:

$$J_z\varepsilon = M_z^e. \tag{16.15}$$

Equation (16.15) shows that, for a given torque M_z^e, the greater the moment of inertia of a body, the less the angular velocity, and vice versa. Thus, we see that in rotational motion the moment of inertia of a body actually plays the same role as mass in translational motion, i.e., it is the measure of a body's inertia in rotational motion.

Note the following special cases:

(1) If M_z^e=0, ω=const, i.e., the rotation is uniform;

(2) If M_z^e=const, ε=const., i.e., the rotation is uniformly variable.

Eq. (16.14) is analogous in form to the differential equation of rectilinear motion of a particle; therefore, the methods of integration are also analogous.

6. Plane Motion of a Rigid Body

The position of a body performing plane motion is specified at any instant by the position of any pole and the angle of rotation of the body about that pole. Dynamical problems are much more simple solved if the centre of mass C of a body is taken as the pole and the position of the body is defined by coordinates x_C, y_C, and angle φ (the body is depicted as intersected by a plane parallel to the plane of motion and passing through point C).

Let there be acting on the body a coplanar system of external forces $\mathbf{F}_1^e$, $\mathbf{F}_2^e$, ..., $\mathbf{F}_n^e$. The equation of motion of point C can be found from the theorem of the motion of centre of mass:

$$m\boldsymbol{a}_C = \sum_k \mathbf{F}_k^e, \tag{16.16}$$

and the rotation about C is given by Eq. (16.14), since the theorem from which it was derived is also valid for the motion of a system about the centre of mass. Finally, after projecting both sides of Eq. (16.16) on the coordinate axes, we obtain:

$$ma_{Cx} = \sum_k F_{kx}^e, \quad ma_{Cy} = \sum_k F_{ky}^e, \quad J_C\varepsilon = \sum_k m_C(\mathbf{F}_{kz}^e), \tag{16.17}$$

or

$$m\frac{d^2x_C}{dt^2} = \sum_k F_{kx}^e, \quad m\frac{d^2y_C}{dt^2} = \sum_k F_{ky}^e, \quad J_C\frac{d^2\varphi}{dt^2} = \sum_k m_C(\mathbf{F}_{kz}^e). \tag{16.18}$$

Eqs. (16.18) are the *differential equations of plane motion of a rigid body*. With their help we can develop the equation of motion of a body if the forces are given or we can determine the principal vector and principal moment of the acting forces if the law of motion is known.

LECTURE 17
D'ALEMBERT'S PRINCIPLE

All the methods of solving the problems of dynamics examined up till now were based on equations derived either directly from Newton's laws or from the general theorems, which are corollaries of those laws. However, the equations of motion or equilibrium conditions of a mechanical system can also be obtained on the basis of other general propositions called the *principles of mechanics*. We shall see that in many cases application of those principles offers better methods of problem solutions. In this chapter we shall examine one of the general principles of mechanics known as *D'Alembert's principle*.

1. D'Alembert's principle

Let there be a system of n material particles. Selecting any particle of mass m_k, assume it to be acted upon by external and internal forces $\mathbf{F}_k^e$ and $\mathbf{F}_k^i$ (which include both active forces and the reactions of constraints), which impart it an acceleration $\boldsymbol{a}_k$ with respect to an inertial reference frame.

Let us introduce the quantity

$$\mathbf{\Phi}_k = -m_k \boldsymbol{a}_k , \tag{17.1}$$

with the dimension of force. The vector quantity equal in magnitude to the product of the particle's mass and acceleration and directed in the opposite sense of the acceleration is called the *force of inertia* of that particle (sometimes the *D'Alembert inertia force*).

Motion of a particle, we then find, satisfies the following **D'Alembert's principle for a material particle**: *If, at any moment of time, to the effective forces* $\mathbf{F}_k^e$ *and* $\mathbf{F}_k^i$ *acting on the particle is added the inertia force* $\mathbf{\Phi}_k$, *the resultant force system will be in equilibrium*, i.e.,

$$\mathbf{\Phi}_k + \mathbf{F}_k^e + \mathbf{F}_k^i = 0 , \tag{17.2}$$

It will be readily observed that D'Alembert's principle is equivalent to Newton's second law, and vice versa. For Newton's second law gives for this particle

$$m_k \boldsymbol{a}_k = \mathbf{F}_k^e + \mathbf{F}_k^i .$$

Transferring $m_k \boldsymbol{a}_k$ to the right-hand side of the equation, and taking into account the notation (17.1), we arrive at Eq. (17.2). Conversely, by transferring $\mathbf{F}_k^i$ to the other side of Eq. (17.2), and taking into account (17.1), we obtain the formula expressing Newton's second law.

Reasoning similarly for all the particles of the system, we arrive at the following result, which expresses **D'Alembert's principle for a system**: *If, at any moment of time, to the effective external and internal forces acting on every particle of a system are added the respective inertia forces, the resultant force system will be in equilibrium, and the equations of statics will apply to it*.

Mathematically D'Alembert's principle is expressed by a set of n simultaneous vector equations of the form (17.2) which, apparently, are equivalent to the differential equations of motion of a system.

The value of D'Alembert's principle is that, when directly applied to problems of dynamics, the equations of motion of a system can be written in the form of the well-known equations of equilibrium; this makes for uniformity in the approach to problem solutions and usually greatly simplifies the computations. Furthermore, when used in conjunction with the principle of virtual displacement, which will be examined in the following chapter, D'Alembert's principle yields a new general method of solution of problems of dynamics.

In applying D'Alembert's principle it should be remembered that, like the fundamental law of dynamics, it refers to motion considered with respect to an inertial frame of reference. That means that acting on the particles of the mechanical system whose motion is being investigated are only the external and internal forces $\mathbf{F}_k^e$ and $\mathbf{F}_k^i$ that appear as a consequence of the interactions of the particles of the system among themselves and with bodies not belonging to the system; it is under the action of those forces that the particles of the system are moving with their respective accelerations $\boldsymbol{a}_k$. The inertia forces mentioned in D'Alembert's principle do not act on the moving particles [otherwise, by Eqs. (17.2), the points would be at rest or in uniform motion in which case, as is apparent from Eq. (17.1), there would be no inertia forces]. The introduction of inertia forces is but a device making it possible to examine the equations of dynamics by the simpler methods of statics.

We know from statics that the geometrical sum of balanced forces and the sum of their moments with respect to any centre O are zero; we know, further, from the principle of solidification that this holds not only for forces acting on a rigid body but for any deformable system. Thus, according to D'Alembert's principle, we must have:

$$\begin{gathered}\sum_k \left(\mathbf{\Phi}_k + \mathbf{F}_k^e + \mathbf{F}_k^i\right) = 0 \\ \sum_k \left(\mathbf{m}_O(\mathbf{\Phi}_k) + \mathbf{m}_O(\mathbf{F}_k^e) + \mathbf{m}_O(\mathbf{F}_k^i)\right) = 0\end{gathered}, \qquad (17.3)$$

Let us introduce the following notation:

$$\sum_k \mathbf{\Phi}_k = \mathbf{\Phi}, \quad \mathbf{M}_O^{\Phi} = \sum_k \mathbf{m}_O(\mathbf{\Phi}_k), \qquad (17.4)$$

The quantities $\mathbf{\Phi}$ and $\mathbf{M}_O^{\Phi}$ are respectively the principal *vector of the inertia forces* and their *principal moment with respect to a centre O*. Taking into account that the sum of the internal forces and the sum of their moments are each zero, we obtain:

$$\sum_k \mathbf{F}_k^e + \mathbf{\Phi} = 0, \quad \sum_k \mathbf{m}_O(\mathbf{F}_k^e) + \mathbf{M}_O^{\Phi} = 0. \qquad (17.5)$$

The use of Eqs. (17.5), which follow from D'Alembert's principle, simplifies the process of problem solution because the equations do not contain the internal forces. Actually, Eqs. (17.5) are equivalent to the equations expressing the theorems of the change in the momentum and the total angular momentum of a system, differing from them only in form.

Eqs. (17.5) are especially convenient in investigating the motion of a rigid body or a system of rigid bodies. For the complete investigation of any deformable system these equations, however, are insufficient.

For the projections on a set of coordinate axes, Eqs. (17.5) give equations analogous to the corresponding equations of statics. To use these equations for solving problems we must know the principal vector and the principal moment of the inertia forces.

2. The principal vector and the principal moment of the inertia forces of a rigid body

It follows from Eqs. (17.4) that a system of inertia forces applied to a rigid body can be replaced by a single force equal to **Φ** and applied at the centre O, and a couple of moment $\mathbf{M}_O^{\Phi}$. The principal vector of a system, it will be recalled, does not depend on the centre of reduction and can be computed at once. Taking into account Eq. (17.1), we will have:

$$\mathbf{\Phi} = -\sum_k m_k \boldsymbol{a}_k = -M\boldsymbol{a}_C. \tag{17.6}$$

Thus, *the principal vector of the inertia forces of a moving body is equal to the product of the mass of the body and the acceleration of its centre of mass, and is opposite in direction to the acceleration*.

If we resolve the acceleration $\boldsymbol{a}_C$ into its tangential and normal components, then vector **Φ** will resolve into components

$$\mathbf{\Phi}_\tau = -M\boldsymbol{a}_C^\tau, \quad \mathbf{\Phi}_n = -M\boldsymbol{a}_C^n. \tag{17.7}$$

Let us determine the principal moment of the inertia forces for particular types of motion.

(1) Translational motion. In this case a body has no rotation about its centre of mass C, from which we conclude that

$$\sum_k \mathbf{m}_O(\mathbf{F}_k^e) = 0,$$

and Eq. (17.5) gives M_C=0.

Thus, *in translational motion, the inertia forces of a rigid body can be reduced to a single resultant* **Φ** *through the centre of mass of the body*.

(2) Plane motion. Let a body have a plane of symmetry, and let it be moving parallel to the plane. By virtue of symmetry, the principal vector and the resultant couple of inertia forces lie, together with the centre of mass C, in that plane.

Therefore, placing the centre of reduction in point C, we obtain from Eq. (17.5)

$$\mathbf{M}_O^{\Phi} = -\sum_k \mathbf{m}_O(\mathbf{F}_k^e).$$

On the other hand

$$\sum_k \mathbf{m}_O(\mathbf{F}_k^e) = J_C \varepsilon.$$

We conclude from this that

$$M_C^{\Phi} = -J_C \varepsilon. \qquad (17.8)$$

Thus, in such motion a system of inertia forces can be reduced to a resultant force $\mathbf{\Phi}$ [Eq. (17.6)] applied at the centre of mass C and a couple in the plane of symmetry of the body whose moment is given by Eq. (17.8). The minus sign shows that the moment M_C^{Φ} is in the opposite direction of the angular acceleration of the body.

(3) Rotation about an axis through the centre of mass. Let a body have a plane of symmetry, and let the axis of rotation Cz be normal to the plane through the centre of mass. This case will thus be a particular case of the previous motion. But here $\boldsymbol{a}_C$=0, and consequently, $\mathbf{\Phi}$=0.

Thus, in this case *a system of inertia forces can be reduced to a couple in the plane of symmetry of the body of moment*

$$M_z^{\Phi} = -J_z \varepsilon. \qquad (17.9)$$

In applying Eqs. (17.6) and (17.8) to problem solutions, the magnitudes of the respective quantities are computed and the directions are shown in a diagram.